$$\frac{dL}{dt}=\frac{\partial L}{\partial t}+\sum_{\alpha=1}^{s}\frac{\partial L}{\partial q_\alpha}\dot{q}_\alpha+\sum_{\alpha=1}^{s}\frac{\partial L}{\partial \dot{q}_\alpha}\frac{d\dot{q}_\alpha}{dt}$$

在主动力全是保守力的情况下，

利用完整系统的拉格朗日方程 $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha}-\frac{\partial L}{\partial q_\alpha}=0\ (\alpha=1,2,\cdots,s)$

把 $\frac{\partial L}{\partial q_\alpha}$ 改写，即得

$$\frac{dL}{dt}=\frac{\partial L}{\partial t}+\sum_{\alpha=1}^{s}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)\dot{q}_\alpha+\sum_{\alpha=1}^{s}\frac{\partial L}{\partial \dot{q}_\alpha}\frac{d\dot{q}_\alpha}{dt}$$
$$=\frac{\partial L}{\partial t}+\frac{d}{dt}\left(\sum_{\alpha=1}^{s}\frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha\right)$$

$\frac{\partial L}{\partial \dot{q}_\alpha}$ 就是广义动量 p_α，这样 $\frac{d}{dt}\left(\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-L\right)=-\frac{\partial L}{\partial t}$

定义广义能量函数 $H=\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-L=\sum_{\alpha=1}^{s}\frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha-L$

则 $\frac{dH}{dt}=-\frac{\partial L}{\partial t}$

若 L 不是时间的显函数，$L=L(p,\dot{q})$，即 $\frac{\partial L}{\partial t}=0$.

则有广义能量积分（或称雅可比积分）$H=const$

势能 V 是与广义速度无关的，因此 $\frac{\partial L}{\partial \dot{q}_\alpha}\rightarrow\frac{\partial T}{\partial \dot{q}_\alpha}$

设变换式 $\vec{r}_i=\vec{r}_i(q)$ 不显含时间，

即 $\frac{\partial \vec{r}_i}{\partial t}=0$，则 $\dot{\vec{r}}_i=\sum_{\alpha=1}^{s}\frac{\partial \vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha$，于是

$$T=\sum_{i=1}^{n}\frac{1}{2}m_i\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i=\sum_{i=1}^{n}\frac{1}{2}m_i\sum_{\alpha=1}^{s}\dot{q}_\alpha\frac{\partial \vec{r}_i}{\partial q_\alpha}\cdot\sum_{\beta=1}^{s}\frac{\partial \vec{r}_i}{\partial q_\beta}\dot{q}_\beta$$
$$=\sum_{i=1}^{n}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\frac{1}{2}m_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\cdot\frac{\partial \vec{r}_i}{\partial q_\beta}\dot{q}_\alpha\dot{q}_\beta$$

齐次函数的欧拉定理

对于变量 $x_1,x_2,\cdots,x_n$ 的 m 次齐次多项式 $f(x_1,x_2,\cdots,x_n)$

$$f(\lambda x_1,\lambda x_2,\cdots,\lambda x_n)=\lambda^m f(x_1,x_2,\cdots,x_n)$$

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}x_i=mf$$

$\frac{\partial}{\partial \dot{q}_r}\dot{\vec{r}}_i \quad \frac{\partial}{\partial \dot{q}_i}\dot{\vec{r}}_i$　$\dot{\vec{r}}_i\rightarrow\sum_{\alpha=1}^{s}\frac{\partial \vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha$

$\uparrow \quad \uparrow$

$\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i$

L关系 $\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_r}$

$$\frac{\partial T}{\partial \dot{q}_r}=\sum_{i=1}^{n}\sum_{\alpha=1}^{s}\frac{1}{2}m_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\cdot\frac{\partial \vec{r}_i}{\partial q_r}\dot{q}_\alpha+\sum_{i=1}^{n}\sum_{\beta=1}^{s}\frac{1}{2}m_i\frac{\partial \vec{r}_i}{\partial q_r}\cdot\frac{\partial \vec{r}_i}{\partial q_\beta}\dot{q}_\beta$$
$$=\sum_{i=1}^{n}\sum_{\alpha=1}^{s}m_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\cdot\frac{\partial \vec{r}_i}{\partial q_r}\dot{q}_\alpha$$，于是

$$\sum_{r=1}^{s}\frac{\partial T}{\partial \dot{q}_r}\dot{q}_r=\sum_{i=1}^{n}\sum_{\alpha=1}^{s}\sum_{r=1}^{s}m_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\cdot\frac{\partial \vec{r}_i}{\partial q_r}\dot{q}_\alpha\dot{q}_r=2T$$

如果约束是非定常的，则变换式 $\vec{r}_i=\vec{r}_i(q,t)$ 难免显含时间

即使约束是稳定的，也可能由于选择了

某些广义坐标（例如平移坐标系）

变换式 $\vec{r}_i=\vec{r}_i(q,t)$ 显含时间 t.

$$\dot{\vec{r}}_i=\frac{\partial \vec{r}_i}{\partial t}+\sum_{\alpha=1}^{s}\frac{\partial \vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha$$

$$T=\sum_{i=1}^{n}\frac{1}{2}m_i(\quad)\cdot(\quad)$$

$$=\sum_{i=1}^{n}\left\{\frac{1}{2}m_i\left(\frac{\partial \vec{r}_i}{\partial t}\right)^2+\sum_{\alpha=1}^{s}m_i\frac{\partial \vec{r}_i}{\partial t}\cdot\frac{\partial \vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha+\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\frac{1}{2}m_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\cdot\frac{\partial \vec{r}_i}{\partial q_\beta}\dot{q}_\alpha\dot{q}_\beta\right\}$$

$$=T_0+T_1+T_2$$

$$\sum_{\alpha=1}^{s}\frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha=0T_0+1T_1+2T_2=T_1+2T_2$$

$$H=\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-L=(T_1+2T_2)-(T_0+T_1+T_2-V)=T_2-T_0+V.$$

广义能量