正则变换

在哈密顿动力学中，广义坐标 $q_\alpha\ (\alpha=1,2,\cdots,s)$ 和广义动量 $p_\alpha\ (\alpha=1,2,\cdots,s)$ 变换：

$$\begin{cases} P_\alpha = P_\alpha(p,q,t) \\ Q_\alpha = Q_\alpha(p,q,t) \end{cases} (\alpha=1,2,\cdots,s)$$

要求变换后的动力学方程仍然是哈密顿正则方程 → 正则变换. 如何满足？

$$\delta\int_{t_1}^{t_2}\left[\sum_{\alpha=1}^{s} p_\alpha \dot q_\alpha - H(p,q,t)\right]dt \underset{\downarrow}{\xLeftrightarrow{\text{等价}}} \delta\int_{t_1}^{t_2}\left[\sum_{\alpha=1}^{s} P_\alpha \dot Q_\alpha - K(P,Q,t)\right]dt=0$$

可以相差某个函数 U 对时间的全导数

$$\left(\sum_{\alpha=1}^{s} p_\alpha \dot q_\alpha - H\right)-\left(\sum_{\alpha=1}^{s} P_\alpha \dot Q_\alpha - K\right)=\frac{dU}{dt} \quad 即 \quad \sum_{\alpha=1}^{s} p_\alpha dq_\alpha-\sum_{\alpha=1}^{s} P_\alpha dQ_\alpha+(K-H)\,dt=dU$$

哈密顿-雅可比方程

正则变换的目标：若 $K(P,Q,t)=0$，则保证所有正则变量都是常数

$$\begin{cases} \dot P_\alpha = -\dfrac{\partial K}{\partial Q_\alpha}=0 \\ \dot Q_\alpha = \dfrac{\partial K}{\partial P_\alpha}=0 \end{cases}$$

母函数

若用 $U_1=U_1(q,Q,t)$，则参照 $\sum_{\alpha=1}^{s} p_\alpha dq_\alpha-\sum_{\alpha=1}^{s} P_\alpha dQ_\alpha+(K-H)\,dt=dU$，有

$$\begin{cases} p_\alpha=\dfrac{\partial U_1}{\partial q_\alpha}, \quad P_\alpha=-\dfrac{\partial U_1}{\partial Q_\alpha} \quad (\alpha=1,2,\cdots,s) \\ K-H=\dfrac{\partial U_1}{\partial t} \end{cases}$$

若用 $U_3(q,P,t)=U_1(q,Q,t)+\sum_{\alpha=1}^{s} P_\alpha Q_\alpha$（勒让德变换），则

$$dU_3=dU_1+\sum_{\alpha=1}^{s} P_\alpha dQ_\alpha+\sum_{\alpha=1}^{s} Q_\alpha dP_\alpha$$

$$=\sum_{\alpha=1}^{s} p_\alpha dq_\alpha+\sum_{\alpha=1}^{s} Q_\alpha dP_\alpha+(K-H)\,dt \quad 有$$

$$\begin{cases} p_\alpha=\dfrac{\partial U_3}{\partial q_\alpha}, \quad Q_\alpha=\dfrac{\partial U_3}{\partial P_\alpha} \quad (\alpha=1,2,\cdots,s), \\ K-H=\dfrac{\partial U_3}{\partial t} \end{cases}$$

哈密顿特征函数

由上面 $K-H=\frac{\partial U_3}{\partial t}$，若 $K=0$，则

$$H(q,p,t)+\frac{\partial U_3}{\partial t}=0 \Rightarrow H\left(q_1,q_2,\cdots,q_s,\frac{\partial U_3}{\partial q_1},\frac{\partial U_3}{\partial q_2},\cdots,\frac{\partial U_3}{\partial q_s},t\right)+\frac{\partial U_3}{\partial t}=0.$$

→ 哈密顿-雅可比方程、U_3 的解为哈密顿主函数 $S(q,P,t)$.

$$\frac{dS}{dt}=\sum_{\alpha=1}^{s}\frac{\partial S}{\partial q_\alpha}\dot q_\alpha+\sum_{\alpha=1}^{s}\frac{\partial S}{\partial P_\alpha}\dot P_\alpha+\frac{\partial S}{\partial t}$$

$$=\sum_{\alpha=1}^{s} p_\alpha \dot q_\alpha+0+(0-H)=L$$

由此 $S=\int L\,dt$.

若 H 不显含时间，则可以把哈密顿-雅可比方程中的 q 与 t 分离.

令 $S(q,P,t)=W(q,P)+f(t)$，则 $H\left(q,\frac{\partial W}{\partial q}\right)=-f(t)$. 分解为

$$\begin{cases} f'(t) = -E \\ H\left(q_1, q_2, \cdots, q_s, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}, \cdots, \frac{\partial W}{\partial q_s}\right) = E \end{cases} \quad \begin{array}{l} \Rightarrow f(t) = -Et \\ \Rightarrow K = H + \frac{\partial W}{\partial t} = H = E \end{array}$$

$W(q, P)$ 叫哈密顿特征函数.

例：用哈密顿-雅可比方程求解谐振子问题. $H = \frac{1}{2m}P^2 + \frac{1}{2}kx^2$

解：$H = \frac{1}{2m}p^2 + \frac{1}{2}kx^2 \Rightarrow \frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}kx^2 = E.$

$$\Rightarrow W(x, E) = \sqrt{mk}\int\sqrt{\frac{2E}{k} - x^2}\, dx$$

变换后的"P".

解一：$\dot{X} = \frac{\partial K}{\partial P} = \frac{\partial K}{\partial E} = 1 \Rightarrow X = t - t_0$

$$X = \frac{\partial W}{\partial P} = \frac{\partial W}{\partial E} = \sqrt{\frac{m}{k}}\int\frac{dx}{\sqrt{\frac{2E}{k} - x^2}}$$

$$= \sqrt{\frac{m}{k}}\arcsin\left(x\sqrt{\frac{k}{2E}}\right)$$

$$x = \sqrt{\frac{2E}{k}}\sin\left(\sqrt{\frac{k}{m}}X\right)$$

$$= \sqrt{\frac{2E}{k}}\sin\left(\sqrt{\frac{k}{m}}t - \sqrt{\frac{k}{m}}t_0\right)$$

解二：$S(x, E, t) = W(x, E) - Et$

正则变换的母函数

$$= \sqrt{mk}\int\sqrt{\frac{2E}{k} - x^2}\, dx - Et$$

$$K = H + \frac{\partial S}{\partial t} = E - E = 0 \Rightarrow X = -t_0$$

变换公式：

$$X = \frac{\partial S}{\partial P} = \frac{\partial S}{\partial E} = \sqrt{\frac{m}{k}}\int\frac{dx}{\sqrt{2E/k - x^2}} - t$$

$$= \sqrt{\frac{m}{k}}\arcsin\left(x\frac{k}{2E}\right) - t$$

$$x = \sqrt{\frac{2E}{k}}\sin\left(\sqrt{\frac{k}{m}}t + \sqrt{\frac{k}{m}}X\right)$$

得 $x = \sqrt{\frac{2E}{k}}\sin\left(\sqrt{\frac{k}{m}}t - \sqrt{\frac{k}{m}}t_0\right)$

讨论：U_1 与 U_3 形式的母函数

$$W(x, E) = \sqrt{mk}\int\sqrt{\frac{2E}{k} - x^2}\, dx$$

$$= \frac{1}{2}\sqrt{mk}\, x\sqrt{\frac{2E}{k} - x^2} + E\sqrt{\frac{m}{k}}\arcsin\left(\sqrt{\frac{k}{2E}}x\right)$$

是 U_3 形式的母函数

$$= \frac{1}{2}\sqrt{mk}\, x\sqrt{\frac{2E}{k} - x^2} + EX(x, E)$$

$$U_1(x, X) = W(x, E(x)) - EX$$

$$= \frac{1}{2}\sqrt{mk}\, x\sqrt{\frac{2E}{k} - x^2} \qquad \text{考虑 } X = \sqrt{\frac{m}{k}}\arcsin\left(x\frac{k}{2E}\right)$$

$$= \frac{1}{2}\sqrt{mk}\, x^2\cot\left(\sqrt{\frac{k}{m}}X\right)$$

正则微扰

现考虑原系统可精确求解，又对系统加一个微小的、不含时的微扰 $H=H_0+\varepsilon H'$

如果 $\varepsilon=0$，则可按哈密顿-雅可比方程求解，

$$\frac{\partial S}{\partial t}+H_0\left(q_1,q_2,\cdots,q_s;\frac{\partial S}{\partial q_1},\frac{\partial S}{\partial q_2},\cdots,\frac{\partial S}{\partial q_s};t\right)=0$$

有一个全积分 $S(q_1,q_2,\cdots,q_s;\alpha_1,\alpha_2,\cdots,\alpha_s;t)$。系统的解：

$$\begin{cases}\dfrac{\partial S}{\partial \alpha_\alpha}=\beta_\alpha \\ \dfrac{\partial S}{\partial q_\alpha}=p_\alpha\end{cases}$$

其中 $\alpha_1,\alpha_2,\cdots,\alpha_s,\beta_1,\beta_2,\cdots,\beta_s$ 是不变的积分常数。

现考虑微扰，$\alpha_1,\alpha_2,\cdots,\alpha_s,\beta_1,\beta_2,\cdots,\beta_s$ 不再是常数，但仍是正则变量。

$K=H_0+\varepsilon H'+\frac{\partial S}{\partial t}=\varepsilon H'$。于是新正则方程

$$\begin{cases}\dot\alpha_\alpha=-\dfrac{\partial(\varepsilon H')}{\partial \beta_\alpha} \\ \dot\beta_\alpha=\dfrac{\partial(\varepsilon H')}{\partial \alpha_\alpha}\end{cases}$$

称为微扰方程。可用级数展开法求近似解

例：非线性振动，势能 $V=\frac{1}{2}kx^2+\frac{1}{4}bx^4$。$H=\frac{p^2}{2m}+\frac{1}{2}kx^2+\frac{1}{4}bx^4=H_0+\frac{1}{4}bx^4$。其中 $H_0=\frac{p^2}{2m}+\frac{1}{2}kx^2$

解：H_0 对应的无微扰解：

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2+\frac{1}{2}kx^2=0$$ 于是 $S=-\alpha t+\sqrt{2m}\int\sqrt{\alpha-\frac{1}{2}kx^2}\,dx$.

$$\begin{cases}p=\dfrac{\partial S}{\partial x}=\sqrt{2m}\sqrt{\alpha-\frac{1}{2}kx^2} \\ \beta=\dfrac{\partial S}{\partial \alpha}=-t+\dfrac{1}{\omega_0}\arcsin\dfrac{x}{\sqrt{\frac{2\alpha}{k}}}\end{cases}\Rightarrow\begin{cases}x=\sqrt{\dfrac{2\alpha}{k}}\sin(\omega_0(t+\beta)) \\ p=\sqrt{2m\alpha}\cos(\omega_0(t+\beta))\end{cases}$$ 其中 $\omega_0=\sqrt{\frac{k}{m}}$.

现在考虑微扰，α 和 β 不再是常数。

$$\varepsilon H'=\frac{1}{4}bx^4=\frac{\alpha^2 b}{k^2}\sin^4(\omega_0(t+\beta))=\frac{\alpha^2 b}{8k^2}[3-4\cos(2\omega_0(t+\beta))+\cos(4\omega_0(t+\beta))]$$

$$\begin{cases}\dot\beta=\dfrac{\partial(\varepsilon H')}{\partial\alpha} & \Rightarrow \dot\beta=\dfrac{\alpha b}{4k^2}[3-4\cos(2\omega_0(t+\beta))+\cos(4\omega_0(t+\beta))] \\ \dot\alpha=-\dfrac{\partial(\varepsilon H')}{\partial\beta} & \Rightarrow \dot\alpha=-\dfrac{\alpha^2 b\omega_0}{2k^2}[2\sin(2\omega_0(t+\beta))-\sin(4\omega_0(t+\beta))]\end{cases}$$

保留 α 项，忽略 α^2 项.

$$\begin{cases}\alpha_1=\alpha_0=\text{const} \\ \beta_1=\dfrac{\alpha_0 b}{4k^2}\displaystyle\int[3-4\cos(2\omega_0(t+\beta))+\cos(4\omega_0(t+\beta))]dt.\end{cases}$$

为方便，假定 $t=0$ 时 $x=0$. $\Rightarrow$ $t=0$ 时 $\beta=0$.

$$\Rightarrow \beta_1=\frac{\alpha_0 b}{4k^2}\left[3t-\frac{2}{\omega_0}\sin(2\omega_0(t+\beta))+\frac{1}{4\omega_0}\sin(4\omega_0(t+\beta))\right]$$

长时间中，只有第一项起主要作用 $\Rightarrow \beta_1=\frac{3}{4}\frac{\alpha_0 b}{k^2}t$.

$$x=\sqrt{\frac{2\alpha_0}{k}}\sin\left(\omega_0\left(1+\frac{3}{4}\frac{\alpha_0 b}{k^2}\right)t\right)$$

则在一级近似下，频率修正为 $\omega_1=\omega_0\left(1+\frac{3}{4}\frac{\alpha_0 b}{k^2}\right)$

考虑到振幅 $A=\sqrt{\frac{2\alpha}{k}}\approx\sqrt{\frac{2\alpha_0}{k}}$，$\omega_1=\omega_0\left(1+\frac{3}{8}\frac{b}{k}A^2\right)$.

可分离系统

如果系统的所有变量均可分离，就是完全可分离系统，以H不显含时间为例.

如果 H中 q_1 以及与之对应的偏微商 $\frac{\partial W}{\partial q_1}$ 以某种组合的形式 $\phi_1(q_1, \frac{\partial W}{\partial q_1})$ 出现，ϕ_1 中无 q_α 和 $\frac{\partial W}{\partial q_\alpha}$ $(\alpha\neq 1)$，即

$$H\left[q_\alpha, \frac{\partial W}{\partial q_\alpha}, \phi_1(q_1, \frac{\partial W}{\partial q_1})\right] - E = 0$$

设 $W = W'(q_\alpha) + W_1(q_1)$，则 $H\left[q_\alpha, \frac{\partial W}{\partial q_\alpha}, \phi_1(q_1, \frac{dW_1}{dq_1})\right] - E = 0.$

将 ϕ_1 分离出来，$\Phi(q_\alpha, \frac{\partial W'}{\partial q_\alpha}) = \phi_1(q_1, \frac{dW_1}{dq_1})$

恒等成立要求

$$\begin{cases} \phi_1(q_1, \frac{dW_1}{dq_1}) = C_2 \\ H(q_\alpha, \frac{\partial W'}{\partial q_\alpha}, C_2) - E = 0. \end{cases}$$

照此接着分离出所有变量. W 和 H=E 有如下形式

$$\begin{cases} W = \sum_{\alpha=1}^{s} W_\alpha(q_\alpha, E, C_2, \cdots, C_s) \\ \Phi_\alpha(q_\alpha, \frac{dW_\alpha}{dq_\alpha}, E, C_2, \cdots, C_s) = 0 \end{cases} \quad (\alpha = 1, 2, \cdots, s)$$

作用量与角变量

考虑作周期运动的系统，此时，不再取 $E, C_2, \cdots, C_s$ 作为变换后的"动量"，而是取作用量

$$J_\alpha = \oint p_\alpha dq_\alpha = \oint \frac{\partial W_\alpha(q_\alpha, E, C_2, \cdots, C_s)}{\partial q_\alpha} dq_\alpha \quad (\alpha = 1, 2, \cdots, s)$$

J_α 仅是积分常数 $E, C_2, \cdots, C_s$ 的函数 $\Leftrightarrow$ $E, C_2, \cdots, C_s$ 可以用 $J_1, J_2, \cdots, J_s$ 表出，从而

$$W = \sum_{\alpha=1}^{s} W_\alpha(q_\alpha; J_1, J_2, \cdots, J_s)$$

J_α 的共轭变量 w_α 叫做角变量 $w_\alpha = \frac{\partial W}{\partial J_\alpha}$ (无量纲)

$$K = H + \frac{\partial W}{\partial t} = H = E$$

$$\dot{w}_\alpha = \frac{\partial K}{\partial J_\alpha} = \frac{\partial E}{\partial J_\alpha} \Rightarrow w_\alpha = \left(\frac{\partial E}{\partial J_\alpha}\right) t - w_{\alpha 0}.$$

$\frac{\partial E}{\partial J_\alpha}$ 是 q_α 的周期运动频率. $f_\alpha = \frac{\partial E}{\partial J_\alpha}$. 每个自由度频率可不同.

$$\Delta w_\alpha = \oint dw_\alpha = \oint \frac{\partial w_\alpha}{\partial q_\alpha} dq_\alpha = \oint \frac{\partial^2 W}{\partial q_\alpha \partial J_\alpha} dq_\alpha$$

$$= \frac{\partial}{\partial J_\alpha} \oint \frac{\partial W}{\partial q_\alpha} dq_\alpha = \frac{\partial}{\partial J_\alpha} \oint p_\alpha dq_\alpha = \frac{\partial J_\alpha}{\partial J_\alpha} = 1$$

例：求谐振子的频率

解：$H = \frac{1}{2m} p^2 + \frac{1}{2} kx^2 = E \quad \Rightarrow p = \sqrt{mk}\sqrt{\frac{2E}{k} - x^2}$

$$J = \oint p dx = \sqrt{mk} \oint \sqrt{\frac{2E}{k} - x^2}\, dx$$

$$= 2\sqrt{mk} \int_{-\sqrt{2E/k}}^{\sqrt{2E/k}} \sqrt{\frac{2E}{k} - x^2}\, dx = \sqrt{mk}\, x\sqrt{\frac{2E}{k} - x^2} + 2E\sqrt{\frac{m}{k}} \arcsin\left(x\sqrt{\frac{k}{2E}}\right)\Bigg|_{-\sqrt{2E/k}}^{\sqrt{2E/k}}$$

$$= 2\pi E\sqrt{\frac{m}{k}}$$

$$\Updownarrow$$

$E = \frac{1}{2\pi}\sqrt{\frac{k}{m}} J$，于是 $f = \frac{\partial E}{\partial J} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

讨论：正则变换的母函数.

$$W=\frac{1}{2}\sqrt{mk}\,x\sqrt{\frac{2E}{k}-x^2}+E\sqrt{\frac{m}{k}}\arcsin\left(\sqrt{\frac{k}{2E}}\,x\right)$$

$$W(x,J)=\frac{x}{2}\sqrt{km}\sqrt{\frac{J}{\pi\sqrt{mk}}-x^2}+\frac{J}{2\pi}\arcsin\left(x\sqrt{\frac{\pi\sqrt{mk}}{J}}\right)$$ 是U_3形式的母函数

$$w=\frac{\partial W}{\partial J}=\frac{1}{2\pi}\arcsin\left(x\sqrt{\frac{\pi\sqrt{mk}}{J}}\right)$$

$$U_1(x,w)=W(x,J)-Jw$$

$$=\frac{x}{2}\sqrt{km}\sqrt{\frac{J}{\pi\sqrt{mk}}-x^2}+Jw-Jw$$

$$=\frac{x}{2}\sqrt{km}\sqrt{\frac{J}{\pi\sqrt{mk}}-x^2}$$ 考虑 $w=\frac{1}{2\pi}\arcsin\left(x\sqrt{\frac{\pi\sqrt{mk}}{J}}\right)$

$$=\frac{1}{2}\sqrt{mk}\,x^2\cot(2\pi w)$$

浸渐不变量

之前讨论了给系统加一个很小的扰动，现在考虑扰动并不小，但作用很慢的情况.

H含参数λ，在系统运动周期内相对变化很小，即 $T\frac{d\lambda}{dt}\ll\lambda$. ?????

先考虑一个自由度的系统. $H=H(p,q,\lambda)$.

暂且视为$\lambda=\text{const}$，系统作周期运动，$W=W(q,J,\lambda)$.

正则变换：$p=\frac{\partial W}{\partial q}$，$w=\frac{\partial W}{\partial J}$.

$$K=H+\frac{\partial W}{\partial t}=H=E(J,\lambda)$$

现考虑$\lambda=\lambda(t)$，系统不再作周期运动. $W=W(q,J,\lambda(t))$.

$$K(w,J,\lambda(t))=H(q,p,\lambda(t))+\frac{\partial W}{\partial t}$$

$$=E(J,\lambda)+\left(\frac{\partial W}{\partial\lambda}\right)_{q,J}\frac{d\lambda}{dt}$$

$$=E(J,\lambda)+\Lambda\dot\lambda$$ 这里 $\Lambda=\left(\frac{\partial W}{\partial\lambda}\right)_{q,J}$. 其中 $q=q(J,w,\lambda)$

正则方程：

$$\begin{cases}\dot J=-\frac{\partial K}{\partial w}=-\left(\frac{\partial\Lambda}{\partial w}\right)_J\dot\lambda\\ \dot w=\frac{\partial K}{\partial J}=\frac{\partial E}{\partial J}+\left(\frac{\partial\Lambda}{\partial J}\right)_w\dot\lambda\end{cases}$$

一个周期内积分

$$\Delta\Lambda=\oint\frac{\partial\Lambda}{\partial q}dq=\frac{\partial}{\partial\lambda}\oint\frac{\partial W}{\partial q}dq=\frac{\partial}{\partial\lambda}\oint p\,dq=\frac{\partial}{\partial\lambda}J$$ 考虑 $\Lambda=\left(\frac{\partial W}{\partial\lambda}\right)_{q,J}$.

$=0$. $\Rightarrow\Lambda$是q的周期函数.

由于λ变化缓慢，q仍可视为是w的周期函数. 因此Λ和$\left(\frac{\partial\Lambda}{\partial w}\right)_J$都是w的周期函数，周期为1.

将Λ展开为w的傅里叶级数 $\Lambda=\sum_{m=-\infty}^{\infty}a_m(J,\lambda)e^{2\pi imw}$. 考虑 $\int_0^1 e^{2\pi i0w}dw=0$. 所以

$$\overline{\left(\frac{\partial\Lambda}{\partial w}\right)_J}=\int_0^1\left(\frac{\partial\Lambda}{\partial w}\right)_J dw=0$$ (对系统运动周期求平均).

$$\overline{\dot J}=-\overline{\left(\frac{\partial\Lambda}{\partial w}\right)_J\dot\lambda}$$ 考虑周期内λ和$\dot\lambda$都可视为常数.

$$\simeq-\dot\lambda\overline{\left(\frac{\partial\Lambda}{\partial w}\right)_J}=0.$$ $\Rightarrow$ J是浸渐不变量

对于多个自由度的系统同理可证 J_α 是浸渐不变量.

若系统有 n个参数 $\vec{\lambda}=\{\lambda_1,\cdots,\lambda_n\}$, 作代换

$$\lambda \to \vec{\lambda}=\{\lambda_1,\cdots,\lambda_n\},$$

$$\left(\frac{\partial W}{\partial \lambda}\right)_{q,J} \to \left(\frac{\partial W}{\partial \vec{\lambda}}\right)_{q,J}=\left\{\left(\frac{\partial W}{\partial \lambda_1}\right)_{q,J},\cdots,\left(\frac{\partial W}{\partial \lambda_n}\right)_{q,J}\right\},$$

$$\Lambda \to \vec{\Lambda}=\{\Lambda_1,\cdots,\Lambda_n\}$$

含有与λ乘积的项 → 和. e.g., $\left(\frac{\partial W}{\partial \lambda}\right)_{q,J}\frac{d\lambda}{dt} \to \left(\frac{\partial W}{\partial \vec{\lambda}}\right)_{q,J}\frac{d\vec{\lambda}}{dt}=\left(\frac{\partial W}{\partial \lambda_1}\right)_{q,J}\frac{d\lambda_1}{dt}+\cdots+\left(\frac{\partial W}{\partial \lambda_n}\right)_{q,J}\frac{d\lambda_n}{dt}$

仍可得 J_α 是浸渐不变量.

例：谐振子 $k=k(t)$ 随时间缓慢变化

$$W(x,J)=\frac{x}{2}\sqrt{km}\sqrt{\frac{J}{\pi\sqrt{mk}}-x^2}+\frac{J}{2\pi}\arcsin\left(x\sqrt{\frac{\pi\sqrt{mk}}{J}}\right)$$

考虑 $w=\frac{1}{2\pi}\arcsin\left(x\sqrt{\frac{\pi\sqrt{mk}}{J}}\right) \Leftrightarrow \sin(2\pi w)=x\sqrt{\frac{\pi\sqrt{mk}}{J}}$

$$W(x,J)=\frac{x^2}{2}\sqrt{km}\cot\left(2\pi w(x,J,k)\right)+J\,w(x,J,k)$$

$$\Lambda(w,J,k)=\left(\frac{\partial W}{\partial k}\right)_{x,J}=\frac{x^2}{2}\sqrt{m}\left[\left(\frac{1}{2}k^{-\frac{1}{2}}\right)\cot(2\pi w)+\sqrt{k}\left(-\frac{2\pi}{\sin^2(2\pi w)}\right)\frac{\partial w}{\partial k}\right]+J\frac{\partial w}{\partial k}$$

$$=\frac{x^2}{2}\sqrt{m}\frac{1}{2\sqrt{k}}\cot(2\pi w)-\frac{x^2}{2}\sqrt{m}\sqrt{k}(2\pi)\frac{J}{x^2\pi\sqrt{mk}}\frac{\partial w}{\partial k}+J\frac{\partial w}{\partial k}=\frac{x^2}{4}\sqrt{\frac{m}{k}}\cot(2\pi w)$$

$$=\frac{J}{8\pi k}\sin(4\pi w)$$

$$\left(\frac{\partial \Lambda}{\partial w}\right)_J=\frac{J}{2k}\cos(4\pi w)$$

$$\overline{\left(\frac{\partial \Lambda}{\partial w}\right)_J}=\int_0^1\left(\frac{\partial \Lambda}{\partial w}\right)_J dw=\left(\frac{2J}{k}\right)\left(\frac{-1}{4\pi}\right)\sin(4\pi w)\Big|_{w=0}^{1}=0 \Rightarrow \bar{\dot{J}}\simeq\frac{dk}{dt}\left(\frac{\partial \Lambda}{\partial w}\right)_J=0.$$