定理overview2

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变分法

- 对 $I[y]=\int_{x_A}^{x_B} F[y(x),y'(x),x]\,dx$

  y,y',x视为独立

  $\delta I[y]=\int_{x_A}^{x_B}\left(\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'\right)dx$

  $=\int_{x_A}^{x_B}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,dx$ (固定边界条件)

(固定边界条件) 取极值条件 $\delta I=0$.

- 若 δy 任意 $\Rightarrow$ $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$ *

  $\Updownarrow$

  $\Rightarrow$ $\frac{\partial F}{\partial x}+\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}-F\right)=0$ * (

  (若 $\frac{\partial F}{\partial x}=0\Rightarrow y'\frac{\partial F}{\partial y'}-F=\text{const}$)

  多个 $y\Rightarrow$ 广义能量积分 H)

- 多函数、有约束情况 (例)(解题概览)

  平面上封闭曲线长度 l, 求其所围面积最大的曲线方程. 设为D

  1. 设曲线参数方程 $x=x(s)$, $y=y(s)$, $0\le s\le s_0$
  2. $l=\cdots=\int_0^{s_0}F(x,x',y,y',s)\,ds$
  3. $D=\cdots=\int_0^{s_0}G(x,x',y,y',s)\,ds$
  4. 设泛函 $\bar{F}=F(x,x',y,y',s)$

     * $=l+\lambda D$
  5. 代入 Euler

     $\begin{cases}\frac{\partial \bar{F}}{\partial x}-\frac{d}{ds}\frac{\partial \bar{F}}{\partial x'}=0\\ \frac{\partial \bar{F}}{\partial y}-\frac{d}{ds}\frac{\partial \bar{F}}{\partial y'}=0\end{cases}$
  6. 运算(变换)上述方程组, 消λ.
  7. 运算(变换), 得到答案.

Hamilton 原理

- (理想约束)

  $\sum_{i=1}^{N}(F_i-m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i=0$

  $m_i\ddot{\vec{r}}_i\cdot\delta\vec{r}_i=\frac{d}{dt}(m_i\dot{\vec{r}}_i\cdot\delta\vec{r}_i)-\delta\left(\frac{1}{2}m_i\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i\right)$ (认为 m_i 不变)

  $\delta W-\sum_{i=1}^{N}\frac{d}{dt}(m_i\dot{\vec{r}}_i\cdot\delta\vec{r}_i)+\delta T=0$

  $\int_{t_1}^{t_2}(\delta W+\delta T)\,dt-\underbrace{\sum(\quad)}_{=0}=0$ ·(固定边界条件)

  $\Rightarrow\int_{t_1}^{t_2}(\delta W+\delta T)\,dt=0$

  - 若所有力都为有势力 $\delta W=-\delta V$

    $\Rightarrow\int_{t_1}^{t_2}\delta(\underbrace{T-V}_{\text{define }L})=0$

    $\Rightarrow\int_{t_1}^{t_2}\delta L\,dt=0$

    - 对完整系统 变、积换序

      $\Rightarrow\delta\underbrace{\int_{t_1}^{t_2}L\,dt}_{\text{define }S\text{: Hamilton 作用量}}=0$

      $\delta S=0\longrightarrow$ 系统真实运动轨迹 (完整保守) Hamilton

广义动量包含动量、角动量

def $p_\alpha=\frac{\partial L}{\partial \dot{q}_\alpha}$ $(\alpha=1,2,\cdots,s)$

- 若 q_β 反映力学系统整体平移

  平移方向单位矢量为 $\vec{n}$

  $\frac{\partial \vec{r}_i}{\partial q_\beta}=\vec{n}$　$p_\beta=\vec{n}\cdot\left(\sum_{i=1}^{n}m_i\dot{\vec{r}}_i\right)$　$Q_\beta=\sum_{i=1}^{n}\vec{F}_i\cdot\vec{n}$

  $\vec{n}$方向动量分量　主动力之和$\vec{n}$方向分量

- 若 q_β 反映力学系统整体转动

  转轴方向单位矢量为 $\vec{n}$

  $\frac{\partial \vec{r}_i}{\partial q_\beta}=\vec{n}\times\vec{r}_i$　$p_\beta=\vec{n}\cdot\left(\sum_{i=1}^{n}m_i\vec{r}_i\times\dot{\vec{r}}_i\right)$　$Q_\beta=\vec{n}\cdot\left(\sum_{i=1}^{n}\vec{r}_i\times\vec{F}_i\right)$

  $\vec{n}$方向角动量 L 分量　主动力对$\vec{n}$轴力矩

Lagrange 方程

* $\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k}=\frac{\partial \vec{r}_i}{\partial q_k}$　* $\frac{\partial \dot{\vec{r}}_i}{\partial q_k}=\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right)$　Lagrange 关系

$\int_{t_1}^{t_2}(\delta W+\delta T)\,dt=0$

其中 $\delta W=\sum_{k=1}^{n}Q_k\delta q_k$　$\Rightarrow\int_{t_1}^{t_2}\left\{\sum_{k=1}^{n}\left[Q_k+\frac{\partial T}{\partial q_k}-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\right]\delta q_k\right\}dt=0$

$\delta T=\frac{\partial T}{\partial q_k}-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)$ (固定边界)

考虑线性非完整约束: $\sum_{k=1}^{n}a_{lk}dq_k+a_{l0}dt=0$, $l=1,\cdots,m'$

引入 λ_l $(l=1,\cdots,m')$ 未定乘子

$\Rightarrow\int_{t_1}^{t_2}\left\{\sum_{k=1}^{n}\left[Q_k+\frac{\partial T}{\partial q_k}-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)+\sum_{l=1}^{m'}\lambda_l a_{lk}\right]\delta q_k\right\}dt=0$

$\Rightarrow\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}-\frac{\partial T}{\partial q_k}=Q_k+\underbrace{\sum_{l=1}^{m'}\lambda_l a_{lk}}_{\text{define }Q_k'}$　Lagrange

define Q_k' 非完整约束对应广义力

$Q_k=Q_{kc}+Q_{knc}$　$\begin{cases}Q_{kc}=-\frac{\partial V}{\partial q_k}\\ Q_{knc}=\sum_{i=1}^{n}F_{inc}\cdot\frac{\partial \vec{r}_i}{\partial q_k}\end{cases}$

- 设 $V=V(q,t)\Rightarrow\frac{\partial V}{\partial \dot{q}_k}=0$

  $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}-\frac{\partial T}{\partial q_k}=-\frac{\partial V}{\partial q_k}+Q_{knc}+Q_k'$

  $\Rightarrow\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}=Q_{knc}+Q_k'$

  * 理想完整保守系统 $Q_k'=0$ $Q_{knc}=0$

    $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}=0\iff\delta\int_{t_1}^{t_2}L\,dt=0$

    $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}-\frac{\partial T}{\partial q_k}=Q_k$

P.S. 若 $L'=L+\frac{d}{dt}f(q,t)$, 则有相同 Lagrange 方程

广义能量积分

(理想、完整、保守) $\Rightarrow\frac{\partial L}{\partial q_k}=\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}$

$\frac{d}{dt}L=\frac{\partial L}{\partial t}+\sum_{k=1}^{n}\left(\frac{\partial L}{\partial q_k}\dot{q}_k+\frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial t}\right)$

$=\sum_{k=1}^{n}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right)$

$\Rightarrow\frac{d}{dt}\left(\underbrace{\sum_{k=1}^{n}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k-L}_{\text{define }H}\right)=0$

define H 广义能量函数

- $V=V(q,t)$ 时, $\frac{\partial V}{\partial \dot{q}_k}=0$

  $\Rightarrow H=\sum_{k=1}^{n}\left(\frac{\partial T_2}{\partial \dot{q}_k}\dot{q}_k+\frac{\partial T_1}{\partial \dot{q}_k}\dot{q}_k+\frac{\partial T_0}{\partial \dot{q}_k}\dot{q}_k\right)-L$

  $=(2T_2+T_1+T_0)-(T_2+T_1+T_0)$

  $=T_2-T_0+V$　广义能量

  - $\frac{\partial L}{\partial t}=0\Rightarrow H=\text{const}$
  - 定常约束

    可取 q 使 $T_1=T_0=0$

    $\Rightarrow H=T_2+V=T+V$ —— 机械能

广义动量积分

(理想、完整、保守)

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=0$

若 $\frac{\partial L}{\partial q_k}=0\Rightarrow\underbrace{\frac{\partial L}{\partial \dot{q}_k}}_{\text{define }P_k}=\text{Const}$

动能与齐次定理

$T=\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i$　$\dot{\vec{r}}_i=\frac{d\vec{r}_i}{dt}=\frac{\partial \vec{r}_i}{\partial t}+\sum_{k=1}^{n}\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k$

$=\underbrace{\sum_{i=1}^{N}\frac{1}{2}m_i\frac{\partial \vec{r}_i}{\partial t}\cdot\frac{\partial \vec{r}_i}{\partial t}}_{T_0}$

$+\underbrace{\sum_{i=1}^{N}m_i\frac{\partial \vec{r}_i}{\partial t}\sum_{k=1}^{n}\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k}_{T_1}$　define

$+\underbrace{\sum_{i=1}^{N}\frac{1}{2}m_i\left(\sum_{k=1}^{n}\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k\right)\left(\sum_{s=1}^{n}\frac{\partial \vec{r}_i}{\partial q_s}\dot{q}_s\right)}_{T_2}$

诺特定理

封闭系统　变换 $q\to q'=\Phi(q,\varepsilon)$　$L(q,\dot{q},t)$ 不变　$\Phi(q,0)=q$

$\exists$守恒量 $I(q,\dot{q})=\sum_{k=1}^{n}\frac{\partial L}{\partial \dot{q}_k}\frac{d}{d\varepsilon}\Phi(q,\varepsilon)\Big|_{\varepsilon=0}$

例 2　不可伸缩的柔软轻绳绕过两个定滑轮和一个动滑轮 (图 3-3), 滑轮的重量很轻, 质量为 m_1、m_2 和 m_3 的物体分别悬挂于绳的两端和动滑轮下. 求各物体的加速度.

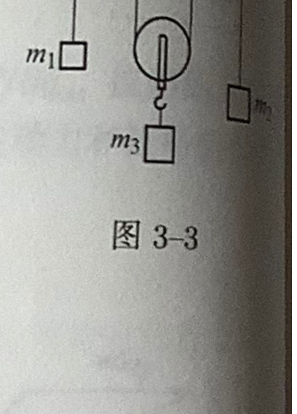

图 3-3

解　三个物体作上下方向的一维运动, 它们的运动又受到一不可伸缩的绳的限制, 因此只有 $3-1=2$ 个自由度.

取左右两边的绳长 l_1 和 l_2 作为力学系统的广义坐标. 因为 $l_1+2l_3+l_2=$ 常数 l. 所以 $l_3=\frac{1}{2}l-\frac{1}{2}(l_1+l_2)$.

主动力是三个物体所受的重力, 它们都是势力. 系统的拉格朗日函数很容易求出:

$$V=-m_1gl_1-m_2gl_2-\frac{1}{2}m_3g\,[l-(l_1+l_2)],$$

$$T=\frac{1}{2}m_1\dot{l}_1^2+\frac{1}{2}m_2\dot{l}_2^2+\frac{1}{2}m_3\left[-\frac{1}{2}\left(\dot{l}_1+\dot{l}_2\right)\right]^2,$$

$$L=T-V=\frac{1}{2}\left(m_1+\frac{1}{4}m_3\right)\dot{l}_1^2+\frac{1}{2}\left(m_2+\frac{1}{4}m_3\right)\dot{l}_2^2+\frac{1}{4}m_3\dot{l}_1\dot{l}_2+\left(m_1-\frac{1}{2}m_3\right)gl_1+\left(m_2-\frac{1}{2}m_3\right)gl_2+\frac{1}{2}m_3gl.$$

拉格朗日方程 (3.1.14) 给出 主动力全是保守力 $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha}-\frac{\partial L}{\partial q_\alpha}=0$ $(\alpha=1,2,\cdots,s)$

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(m_1+\frac{1}{4}m_3\right)\dot{l}_1+\frac{1}{4}m_3\dot{l}_2\right]-\left(m_1-\frac{1}{2}m_3\right)g=0,\\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(m_2+\frac{1}{4}m_3\right)\dot{l}_2+\frac{1}{4}m_3\dot{l}_1\right]-\left(m_2-\frac{1}{2}m_3\right)g=0.\end{cases}$$

由此解得

$$\ddot{l}_1=\frac{4m_1m_2-3m_2m_3+m_1m_3}{4m_1m_2+m_2m_3+m_1m_3}g,$$

$$\ddot{l}_2=\frac{4m_1m_2-3m_1m_3+m_2m_3}{4m_1m_2+m_2m_3+m_1m_3}g,$$

$$\ddot{l}_3=-\frac{1}{2}(\ddot{l}_1+\ddot{l}_2)=\frac{-4m_1m_2+m_2m_3+m_1m_3}{4m_1m_2+m_2m_3+m_1m_3}g.$$