s个自由度　平衡位形 $q_{\alpha 0}$ 记为0

把L在 $q_{\alpha 0}$ 附近展开

$$V = \underbrace{V_0}_{常数设0} + \underbrace{\sum_{\alpha=1}^{s}\left(\frac{\partial V}{\partial q_\alpha}\right)_0}_{平衡位置为0} + \sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\frac{1}{2}\underbrace{\left(\frac{\partial^2 V}{\partial q_\alpha \partial q_\beta}\right)_0}_{记为\ k_{\alpha\beta}=k_{\beta\alpha}} q_\alpha q_\beta + \underbrace{\cdots}_{忽略}$$

$$V = \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} k_{\alpha\beta} q_\alpha q_\beta$$

设 $\vec{r}_i = \vec{r}_i(q)$ 不显含时间(定常约束)

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\vec{r}}_i\cdot\dot{\vec{r}}_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} m_i \frac{\partial \vec{r}_i}{\partial q_\alpha}\cdot\frac{\partial \vec{r}_i}{\partial q_\beta}\dot{q}_\alpha \dot{q}_\beta$$

记 $m_{\alpha\beta} = m_{\beta\alpha} = \sum_{i=1}^{n} m_i \frac{\partial \vec{r}_i}{\partial q_\alpha}\cdot\frac{\partial \vec{r}_i}{\partial q_\beta}$　称为**惯性系数**

$T = \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} m_{\alpha\beta}\dot{q}_\alpha \dot{q}_\beta$　　看作常数(取平衡位形下值)

$$L = \frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\left(m_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta - k_{\alpha\beta}q_\alpha q_\beta\right)$$

根据线性代数理论，总可以变换到一组 ξ_l $(l=1,2,\cdots,s)$

使得 $L = \sum_{l=1}^{s}\left(\frac{1}{2}m_l\dot{\xi}_l^2 - \frac{1}{2}k_l\xi_l^2\right)$　$(l=1,2,3,\cdots,s)$

则 $m_l\ddot{\xi}_l + k_l\xi_l = 0$　$\xi_l = C^{(l)}\cos(\omega_l t + \varphi_l)$　$\left(\omega_l = \sqrt{\frac{k_l}{m_l}}\right)$

但找到 ξ_l 不容易

代入拉格朗日函数中直接求解

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_\alpha}\left(\frac{1}{2}\sum_{\beta=1}^{s}\sum_{\gamma=1}^{s} m_{\beta\gamma}\dot{q}_\beta\dot{q}_\gamma\right) - \frac{\partial}{\partial q_\alpha}\left(-\frac{1}{2}\sum_{\beta=1}^{s}\sum_{\gamma=1}^{s} k_{\beta\gamma}q_\beta q_\gamma\right) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\sum_{\gamma=1}^{s} m_{\alpha\gamma}\dot{q}_\gamma + \frac{1}{2}\sum_{\beta=1}^{s} m_{\beta\alpha}\dot{q}_\beta\right) + \left(\frac{1}{2}\sum_{\gamma=1}^{s} k_{\alpha\gamma}q_\gamma + \frac{1}{2}\sum_{\beta=1}^{s} k_{\beta\alpha}q_\beta\right) = 0$$

$$\sum_{\beta=1}^{s} m_{\alpha\beta}\ddot{q}_\beta + \sum_{\beta=1}^{s} k_{\alpha\beta}q_\beta = 0 \Rightarrow 令\ q_\beta = A_\beta e^{\lambda t}$$

矩阵表述

$T = \frac{1}{2}\tilde{\dot{Q}}M\dot{Q}$　　$V = \frac{1}{2}\tilde{Q}KQ$

其中，$Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_s \end{pmatrix}$　$\dot{Q} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_s \end{pmatrix}$

$M = \begin{pmatrix} m_{11} & \cdots & m_{1s} \\ \vdots & & \vdots \\ m_{s1} & \cdots & m_{ss} \end{pmatrix}$　正定 $M^* = M$　$\tilde{M} = M$　　$K = \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & & \vdots \\ k_{s1} & \cdots & k_{ss} \end{pmatrix}$　正定 $K^* = K$　$\tilde{K} = K$

$$L = \frac{1}{2}\tilde{\dot{Q}}M\dot{Q} - \frac{1}{2}\tilde{Q}KQ$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\right) - \frac{\partial L}{\partial Q} = 0 \Rightarrow M\ddot{Q} + KQ = 0$$

矩阵理论指出，必定存在s个**简正振动模式**

考察某个简正振动模式，其相应的**简正坐标** ξ 不为零而其余的简正坐标均等于零

$$Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_s \end{pmatrix} = \begin{pmatrix} A_1\xi \\ A_2\xi \\ \vdots \\ A_s\xi \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_s \end{pmatrix}\xi = A\xi$$

ξ 满足 $\ddot{\xi} + \omega^2\xi = 0$　ω 为对应**简正角频率**

$MA\ddot{\xi} + KA\xi = 0$

$(-\omega^2 M + K)A = 0$

非零解？

M满秩 $\Rightarrow$ $M^{-1}K\underbrace{A}_{本征矢量} = \overbrace{\omega^2}^{本征值}\underbrace{A}_{本征矢量}$

$|-\omega^2 M + K| = 0$　**特征方程　久期方程**

取某个单根 ω_l^2 代入 $(-\omega^2 M + K)A = 0$　得 $A_1^{(l)} : A_2^{(l)} : \cdots : A_s^{(l)}$

得一个**简正模式**

$$\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_s \end{pmatrix} = \begin{pmatrix} A_1^{(l)} \\ A_2^{(l)} \\ \vdots \\ A_s^{(l)} \end{pmatrix}\xi^{(l)} = \begin{pmatrix} 0 & \cdots & A_1^{(l)} & \cdots & 0 \\ 0 & \cdots & A_2^{(l)} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & A_s^{(l)} & \cdots & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ \xi^{(l)} \\ \vdots \\ 0 \end{pmatrix}$$

如 ω_l^2 为r重根，则可以解出r种比值

$$\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_s \end{pmatrix} = \begin{pmatrix} A_1^{(1)} & A_2^{(1)} & \cdots & A_1^{(s)} \\ A_2^{(1)} & A_2^{(2)} & \cdots & A_2^{(s)} \\ \vdots & \vdots & & \vdots \\ A_s^{(1)} & A_s^{(2)} & \cdots & A_s^{(s)} \end{pmatrix}\begin{pmatrix} \xi^{(1)} \\ \xi^{(2)} \\ \vdots \\ \xi^{(s)} \end{pmatrix}$$

$$Q = A'\xi$$

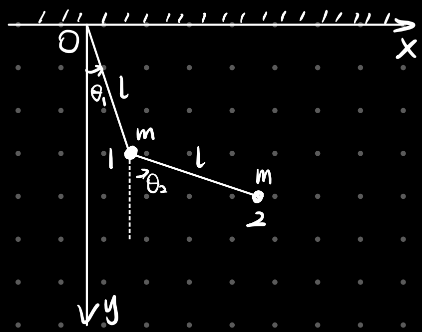
方阵 A'：从简正坐标 ξ_α $(\alpha=1,2,\cdots,s)$ 到(非简正)广义坐标 q_α $(\alpha=1,2,\cdots,s)$ 的变化矩阵

可以证明，采用简正坐标 $\xi^{(l)}$ 表示时，T和V均是平方和的形式

$T = \frac{1}{2}\tilde{\dot{Q}}M\dot{Q} = \frac{1}{2}\tilde{\dot{\xi}}\tilde{A}'MA'\dot{\xi}$　　$V = \frac{1}{2}\tilde{Q}KQ = \frac{1}{2}\tilde{\xi}\tilde{A}'KA'\xi$

$\tilde{A}'MA' = \begin{pmatrix} \mu^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & \mu^{(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \mu^{(s)} \end{pmatrix}$　　$\tilde{A}'KA' = \begin{pmatrix} \mu^{(1)}\omega_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \mu^{(2)}\omega_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \mu^{(s)}\omega_s^2 \end{pmatrix}$

例:



$$T=T_1+T_2$$
$$=\frac{1}{2}ml^2\dot{\theta}_1^2+\frac{1}{2}ml^2(\dot{\theta}_1+\dot{\theta}_2)^2$$
$$=\frac{1}{2}ml^2(2\dot{\theta}_1^2+2\dot{\theta}_1\dot{\theta}_2+\dot{\theta}_2^2)$$

$$V=V_1+V_2$$
$$=-mgl\cos\theta_1-mgl(\cos\theta_1+\cos\theta_2)$$
$$=-mgl(1-\frac{1}{2}\theta_1^2)-mgl(1-\frac{1}{2}\theta_1^2+1-\frac{1}{2}\theta_2^2)$$
$$=-3mgl+mgl(\theta_1^2+\frac{1}{2}\theta_2^2)$$

$$2T=(\dot{\theta}_1\ \dot{\theta}_2)\begin{pmatrix}2ml^2 & ml^2\\ ml^2 & ml^2\end{pmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}\qquad 2V=(\theta_1\ \theta_2)\begin{pmatrix}2mgl & 0\\ 0 & mgl\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$$

$$M=ml^2\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\qquad K=mgl\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$$

$$L=T-V=\frac{1}{2}\tilde{\dot{\theta}}M\dot{\theta}-\frac{1}{2}\tilde{\theta}K\theta \Rightarrow M\ddot{\theta}+K\theta=0$$

令其解为 $\theta_\alpha=A_\alpha e^{i\omega t}$ $(\alpha=1,2)$

$$\begin{vmatrix}2mgl-2ml^2\omega^2 & -ml^2\omega^2\\ -ml^2\omega^2 & mgl-ml^2\omega^2\end{vmatrix}=0$$

$$\begin{cases}\omega_1^2=(2+\sqrt{2})\frac{g}{l}\\ \omega_2^2=(2-\sqrt{2})\frac{g}{l}\end{cases}$$

① $\omega_1^2=(2+\sqrt{2})\frac{g}{l}$

$$\begin{pmatrix}2mgl-2ml^2(2+\sqrt{2})\frac{g}{l} & -ml^2(2+\sqrt{2})\frac{g}{l}\\ -ml^2(2+\sqrt{2})\frac{g}{l} & mgl-ml^2(2+\sqrt{2})\frac{g}{l}\end{pmatrix}\begin{pmatrix}A_1^{(1)}\\ A_2^{(1)}\end{pmatrix}=0$$

$\Rightarrow A_1^{(1)}:A_2^{(1)}=1:-\sqrt{2}$

简正模式: $\theta_1=\xi^{(1)}$, $\theta_2=-\sqrt{2}\xi^{(1)}$

② $\omega_2^2=(2-\sqrt{2})\frac{g}{l}$

$\Rightarrow A_1^{(2)}:A_2^{(2)}=1:\sqrt{2}$

简正模式: $\theta_1=\xi^{(2)}$, $\theta_2=\sqrt{2}\xi^{(2)}$

$$\begin{cases}\theta_1=\xi^{(1)}+\xi^{(2)}\\ \theta_2=-\sqrt{2}\xi^{(1)}+\sqrt{2}\xi^{(2)}\end{cases}\Longleftrightarrow\begin{cases}\xi^{(1)}=\frac{1}{2\sqrt{2}}(\sqrt{2}\theta_1-\theta_2)\\ \xi^{(2)}=\frac{1}{2\sqrt{2}}(\sqrt{2}\theta_1+\theta_2)\end{cases}$$

$$A^{(1)}=A_1^{(1)}\begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix}\qquad A^{(2)}=A_1^{(2)}\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$$

$$\tilde{A}^{(1)}MA^{(2)}=0$$
$$\tilde{A}^{(1)}MA^{(1)}=2(2-\sqrt{2})ml^2(A_1^{(1)})^2=\mu^{(1)}$$
$$\tilde{A}^{(2)}MA^{(2)}=2(2+\sqrt{2})ml^2(A_1^{(2)})^2=\mu^{(2)}$$

$$\tilde{A}'MA'=2ml^2\begin{pmatrix}(2-\sqrt{2})(A_1^{(1)})^2 & 0\\ 0 & (2+\sqrt{2})(A_1^{(2)})^2\end{pmatrix}=\begin{pmatrix}\mu^{(1)} & 0\\ 0 & \mu^{(2)}\end{pmatrix}$$

$$\tilde{A}'KA'=4mgl\begin{pmatrix}(A_1^{(1)})^2 & 0\\ 0 & (A_1^{(2)})^2\end{pmatrix}=\begin{pmatrix}\mu^{(1)}\omega_1^2 & 0\\ 0 & \mu^{(2)}\omega_2^2\end{pmatrix}$$