Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

satisfies $N(t) = N(t-1) + N(t-2)$, where $t \in \mathbb{N}^+$, $t \geq 3$.

The sequence $N(t)$ grows exponentially, that is, $N(t) \approx c\lambda^t$, where λ is the maximum magnitude solution of the characteristic equation $1 - z^{-1} - z^{-2} = 0$.

Solving the characteristic equation yields $(z-\lambda)(z-\beta) = 0$ where $\lambda = \frac{1+\sqrt{5}}{2} > 1$ the Golden Ratio & $\beta = \frac{1-\sqrt{5}}{2} < 1$.

Proof: Let $X(z)$ denote the z-transform of the sequence $N(t)$, $t \geq 1$.

$$X(z) = \sum_{t=1}^{\infty} N(t) z^{-t}$$

$$= 0 \cdot z^{-1} + 1 \cdot z^{-2} + \sum_{t=3}^{\infty} N(t) z^{-t}$$

$$= z^{-2} + \sum_{t=3}^{\infty} \left(N(t-1) + N(t-2)\right) z^{-t}$$

$$= z^{-2} + z^{-1} \sum_{t=2}^{\infty} N(t) z^{-t} + z^{-2} \sum_{t=1}^{\infty} N(t) z^{-t}$$

$$= z^{-2} + z^{-1}\left(X(z) - z^{-1} N(1)\right) + z^{-2} X(z)$$

$$= z^{-2} + z^{-1} X(z) + z^{-2} X(z)$$

$$X(z) = \frac{z^{-2}}{1 - z^{-1} - z^{-2}}$$

$$= \frac{1}{(z-\beta)(z-\lambda)}$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1}{z-\lambda} - \frac{1}{z-\beta} \right)$$

$$= \frac{1}{\sqrt{5}} \frac{1}{z} \left(\frac{1}{1-\frac{\lambda}{z}} - \frac{1}{1-\frac{\beta}{z}} \right)$$

$$= \frac{1}{\sqrt{5} z} \left(\sum_{i=0}^{\infty} \left(\frac{\lambda}{z}\right)^i - \sum_{i=0}^{\infty} \left(\frac{\beta}{z}\right)^i \right) \quad \text{if } z < \frac{\sqrt{5}-1}{2}.$$

$$= \frac{1}{\sqrt{5}} \sum_{i=1}^{\infty} \left(\lambda^{i-1} - \beta^{i-1}\right) z^{-i}$$

$$N(t) = \frac{1}{\sqrt{5}} \left(\lambda^{t-1} - \beta^{t-1}\right)$$

as $n \to \infty$, $|\beta| < 1$, $\beta^{t-1} \to 0$, $N(t) \to \frac{\lambda^{-1}}{\sqrt{5}} \lambda^t$