# 多元函数

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## 极限

白话：以何种方式趋向某点，极限都存在且相同

$f(x,y)=\frac{xy}{x^2+y^2}$ 在 $(x,y)\to(0,0)$ 时无极限

令 $x=\rho\cos\theta_0$，$y=\rho\sin\theta_0$，$(x,y)\to(0,0)\Leftrightarrow \rho\to0^+$

当 $\rho$ 沿 $L: x=\rho\cos\theta_0,\ y=\rho\sin\theta_0$ 趋向于 $(0,0)$ 时

$$\lim_{\substack{x,y\in L\\(x,y)\to(0,0)}} f(x,y)=\lim_{\rho\to0^+}\cos\theta_0\sin\theta_0=\cos\theta_0\sin\theta_0$$

## 偏导

一元函数 可导 $\Rightarrow$ 连续

二元函数 可偏导 $\xRightarrow{\text{各偏导数邻域内有界}}$ 连续

连续 $\not\Rightarrow$ 偏导存在

可微 $\not\Rightarrow$ 连续可微

可微 $\Rightarrow$ 连续

可微 $\Rightarrow$ 可偏导

$$\begin{tikzcd}
\text{连续可微} \arrow[r, shift left] & \text{可微} \arrow[l, shift left, "\times"] \arrow[dl, shift left] \arrow[dr, shift left] & \\
\text{连续} \arrow[ur, shift left, "\times"] \arrow[rr, shift left, "\times"] & & \text{偏导存在} \arrow[ul, shift left, "\times"] \arrow[ll, shift left, "\times"]
\end{tikzcd}$$

## 极值

对各变量是极值点 + 可偏导，偏导为0 $\Rightarrow$ 驻点 $\Longrightarrow$ 极值点

↑ 判别法

设函数 $f(x)=f(x_1,x_2,\cdots,x_m)$ 在点 $a=(a_1,\cdots,a_m)$ 邻近至少是二阶连续可微的。

考查由 $f$ 在点 $a$ 的二阶偏导数组成的方阵

$$H_f(a)=\left(\frac{\partial^2 f}{\partial x_i\partial x_j}(a)\right)_{m\times m}$$

$$=\begin{pmatrix}
\frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_1\partial x_2}(a) & \cdots & \frac{\partial^2 f}{\partial x_1\partial x_m}(a)\\
\frac{\partial^2 f}{\partial x_2\partial x_1}(a) & \frac{\partial^2 f}{\partial x_2^2}(a) & \cdots & \frac{\partial^2 f}{\partial x_2\partial x_m}(a)\\
\vdots & \vdots & & \vdots\\
\frac{\partial^2 f}{\partial x_m\partial x_1}(a) & \frac{\partial^2 f}{\partial x_m\partial x_2}(a) & \cdots & \frac{\partial^2 f}{\partial x_m^2}(a)
\end{pmatrix}$$

正定 $\Rightarrow$ 严格极小

负定 $\Rightarrow$ 严格极大

## 二重积分

累次　　换元

$$\begin{cases}x=x(u,v)\\ y=y(u,v)\end{cases}$$

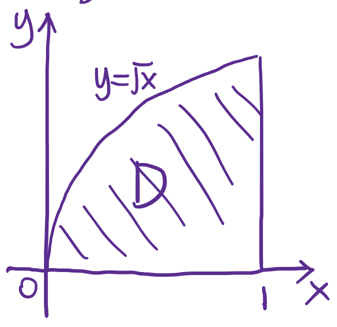
$$\iint_{\varphi(E)} f(x,y)\,d(x,y)=\iint_E f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|d(u,v)$$

极坐标换元

$(x(\rho,\theta),y(\rho,\theta))=(\rho\cos\theta,\rho\sin\theta)$

$$\frac{D(x,y)}{D(\rho,\theta)}=\begin{vmatrix}\cos\theta & -\rho\sin\theta\\ \sin\theta & \rho\cos\theta\end{vmatrix}=\rho\geq0$$

$\iint_D x^2y\,dxdy$




• 先对 y 积

$D=\{(x,y)\mid x\in[0,1],\ 0\leq y\leq\sqrt{x}\}$

$$\begin{aligned}&\iint_D x^2y\,dxdy\\ &=\int_0^1\left(\int_0^{\sqrt{x}}x^2y\,dy\right)dx\\ &=\int_0^1\left(x^2\frac{y^2}{2}\Big|_{y=0}^{y=\sqrt{x}}\right)dx\\ &=\int_0^1 x^2\frac{x}{2}\,dx\\ &=\frac{1}{8}\end{aligned}$$

• 先对 x 积

$D=\{(x,y)\mid y\in[0,1],\ y^2\leq x\leq1\}$

$$\begin{aligned}&\iint_D x^2y\,dxdy\\ &=\int_0^1\left(\int_{y^2}^1 x^2y\,dx\right)dy\\ &=\int_0^1\left(y\frac{x^3}{3}\Big|_{x=y^2}^{x=1}\right)dy\\ &=\int_0^1 y\left(\frac13-\frac{y^6}{3}\right)dy\\ &=\frac{1}{8}\end{aligned}$$

## 三重积分

累次　　换元

$$\begin{cases}x=x(u,v,w)\\ y=y(u,v,w)\\ z=z(u,v,w)\end{cases}$$

$$J(u,v,w)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

柱坐标变换

$x=\rho\cos\theta$

$y=\rho\sin\theta$

$z=z$

$J(\rho,\theta,z)=\rho\geq0$

球坐标变换

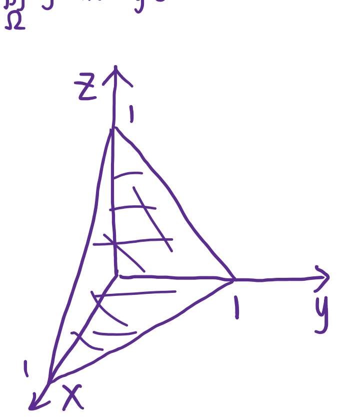
$x=r\sin\theta\cos\varphi$

$y=r\sin\theta\sin\varphi$

$z=r\cos\theta$

$J(r,\theta,\varphi)=r^2\sin\theta\geq0$

$\iiint_\Omega y\,dxdydz$




$D=\{(x,y)\mid 0\leq x\leq 1-y,\ 0\leq y\leq1\}$

$\Omega=\{(x,y,z)\mid (x,y)\in D,\ 0\leq z\leq 1-x-y\}$

$$\iiint_\Omega y\,dxdydz=\iint_D\left(\int_0^{1-x-y}y\,dz\right)dxdy$$

$$=\iint_D y(1-x-y)\,dxdy=\int_0^1\left(\int_0^{1-y}y(1-x-y)\,dx\right)dy$$

$$=\int_0^1\left(-\frac{y}{2}x^2+(y-y^2)x\Big|_{x=0}^{x=y}\right)dy=\int_0^1\frac12 y(1-y^2)\,dy$$

$$=\frac18y^4-\frac13y^3+\frac14y^2\Big|_0^1=\frac{1}{24}$$