曲线 曲面 积分

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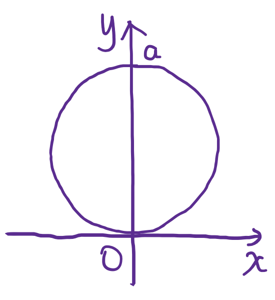
第一类曲线积分

$\int_C \sqrt{x^2+y^2}\, ds \quad C: x^2+y^2=ay$



$(x,y)=(\rho\cos\theta, \rho\sin\theta)$

$x^2+y^2=ay \Longleftrightarrow \rho=a\sin\theta$

选θ作参数 $\theta\in[0,\pi]$

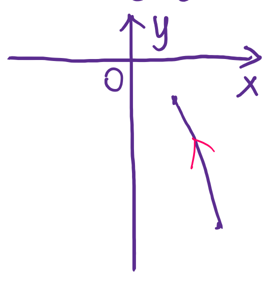
$(x,y)=(a\sin\theta\cos\theta, a\sin^2\theta)$

$$\int_C \sqrt{x^2+y^2}\, ds=\int_0^{\pi}\sqrt{a^2\sin^2\theta}\; a\sqrt{(\cos^2\theta-\sin^2\theta)^2+4\cos^2\theta\sin^2\theta}\, d\theta$$

$$=a^2\int_0^{\pi}\sin\theta\, d\theta=2a^2$$

第二类曲线积分

$\int_C x^2dx+y\,dy \quad C=\{(x,y)\mid x^2+y=0,\ x\in[1,2]\}$



$\vec{r}(x)=(x,-x^2)$

从2到1

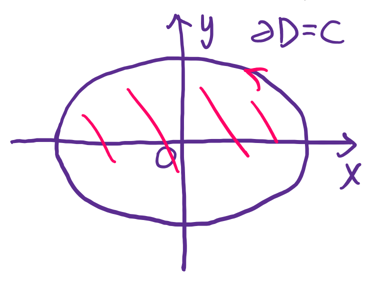
$$\int_C x^2dx+y\,dy=\int_2^1 x^2dx+(-x^2)\,d(-x^2)$$

$$=\int_2^1 x^2dx+2x^3dx=\int_2^1(x^2+2x^3)\,dx$$

$$=-\frac{59}{6}$$

格林公式

$$\oint_C P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$



求椭圆 $x=a\cos\theta,\ y=b\sin\theta$ $(0\le\theta\le 2\pi)$ 所围图形的面积

取 $P(x,y)=-y,\ Q(x,y)=x$

$\oint_C x\,dy=\iint_D\frac{\partial x}{\partial x}dx\,dy=\mathrm{Area}(D)$

$\oint_C(-y)\,dx=\iint_D-\frac{\partial}{\partial y}(-y)\,dx\,dy=\mathrm{Area}(D)$

$A=\frac{1}{2}\oint_C x\,dy-y\,dx$

$=\frac{1}{2}\int_0^{2\pi}a\cos\theta\, d(b\sin\theta)-b\sin\theta\, d(a\cos\theta)$

$=\frac{1}{2}\int_0^{2\pi}ab(\cos^2\theta+\sin^2\theta)\,d\theta$

$=ab\pi$

设ω在U上连续可微，$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$，且U单连通，

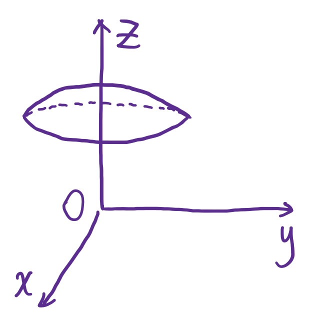
则ω在U上积分与路径无关，亦即有原函数.

—— 检验原函数f是否存在的条件

第一类曲面积分

$\iint_S z\,dS$

$S:\begin{cases}x^2+y^2+z^2=a^2, & a>0\\ z\ge h\end{cases}$



$\begin{cases}x^2+y^2+z^2=a^2\\ z\ge h\end{cases}\Rightarrow\begin{cases}x^2+y^2=a^2-h^2\\ z\ge h\end{cases}$

取 (x,y) 作参数. $z=\sqrt{a^2-x^2-y^2}$

$(x,y)\in D: x^2+y^2\le a^2-h^2$

$\vec{r}(x,y)=(x,y,z(x,y))$

$\frac{\partial\vec{r}}{\partial x}=(1,0,z_x')\quad \frac{\partial\vec{r}}{\partial y}=(0,1,z_y')$

$\left\|\frac{\partial\vec{r}}{\partial x}\times\frac{\partial\vec{r}}{\partial y}\right\|=\sqrt{1+(z_x')^2+(z_y')^2}=\frac{a}{\sqrt{a^2-x^2-y^2}}$

$$\iint_S z\,dS=\iint_D\sqrt{a^2-x^2-y^2}\,\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$$=\iint_D a\,dx\,dy=a\pi(a^2-h^2)$$

第二类曲面积分

参数化：$\vec{r}(u,v)=(x,y,z)\quad \vec{r}: D\to S$

$$\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}=\left(\frac{D(y,z)}{D(u,v)},\frac{D(z,x)}{D(u,v)},\frac{D(x,y)}{D(u,v)}\right)$$

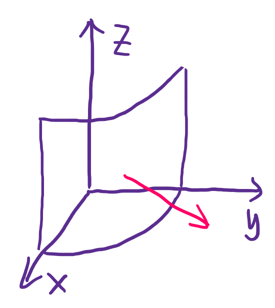
若 $\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}$ 与S定向相同，则保向. 否则反向

保向：$\iint_S P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy$

$$=\iint_D\left(P\frac{D(y,z)}{D(u,v)}+Q\frac{D(z,x)}{D(u,v)}+R\frac{D(x,y)}{D(u,v)}\right)du\,dv$$

反向：$\iint_S=-\iint_D$

$\iint_S z\,dx\,dy+x\,dy\,dz+y\,dz\,dx$



$S:\begin{cases}x^2+y^2=1\\ 0\le z\le 2\\ x\ge 0,\ y\ge 0\end{cases}$

$\vec{r}(\theta,z)=(\cos\theta,\sin\theta,z)$

$(\theta,z)\in D=[0,\frac{\pi}{2}]\times[0,2]$

$\frac{\partial\vec{r}}{\partial\theta}=(-\sin\theta,\cos\theta,0)\quad \frac{\partial\vec{r}}{\partial z}=(0,0,1)$

$\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial z}=(\cos\theta,\sin\theta,0)$

$\theta\in[0,\frac{\pi}{2}]\quad \cos\theta\ge 0$ 前侧 保向

$$\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial z}=\left(\frac{D(y,z)}{D(\theta,z)},\frac{D(z,x)}{D(\theta,z)},\frac{D(x,y)}{D(\theta,z)}\right)$$

$$\iint_S z\,dx\,dy+x\,dy\,dz+y\,dz\,dx$$

$$=\iint_D(z\cdot 0+\cos\theta\cdot\cos\theta+\sin\theta\cdot\sin\theta)\,d\theta\,dz$$

$$=\frac{\pi}{2}\cdot 2=\pi$$

高斯公式

Ω为$\mathbb{R}^3$中有界闭区域，由分片可微曲面S包围，

S定向为外法向，P, Q, R在Ω上连续可微

$$\iint_S P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy$$

$$=\iiint_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

斯托克斯公式

S为定向可微有界曲面，S的边界Γ分段可微.

P, Q, R 在S上有定义，且连续可微，Γ边界正向

$$\int_\Gamma P\,dx+Q\,dy+R\,dz$$

$$=\iint_S\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dy\,dz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dz\,dx+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$$\iint_S\begin{vmatrix}dy\,dz & dz\,dx & dx\,dy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

体三重积分 $\xLeftrightarrow{\text{Gauss公式}}$ 2^{nd} 封闭曲面二重积分

2^{nd} 曲面二重积分 $\xLeftrightarrow{\text{Stokes公式}}$ 2^{nd} 封闭曲线积分