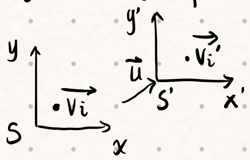
The Center of Mass Frame

König's Theorem   Reduced Mass

König's Theorem

① Suppose there are two frames, S and S'. S' moves with $\boxed{\vec{u}}$ (constant velocity) with respect to S.



a particle moves with $\vec{v}_i$ in S, $\vec{v}_i'$ in S'

$$\vec{v}_i = \vec{u} + \vec{v}_i'$$

Good: conservation of momentum in S
$\Updownarrow$ ✓ true
S'

Bad: conservation of energy in S
$\Updownarrow$ ✗ usually not true
S'

if S' is arbitrary
BUT! The CM frame ...

e.g. for an elastic collision

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$
$$\Updownarrow \checkmark$$
$$m_1(v_{1i}+u) + m_2(v_{2i}+u) = m_1(v_{1f}+u) + m_2(v_{2f}+u)$$

$$\tfrac{1}{2}m_1 v_{1i}^2 + \tfrac{1}{2}m_2 v_{2i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2$$
$$\Updownarrow \times$$
$$\tfrac{1}{2}m_1(v_{1i}+u)^2 + \tfrac{1}{2}m_2(v_{2i}+u)^2 = \tfrac{1}{2}m_1(v_{1f}+u)^2 + \tfrac{1}{2}m_2(v_{2f}+u)^2$$

② Definition of the CM Frame: total momentum of a systems of particles is zero. (☆ important)

another name: zero-momentum frame

We assume S: the "ground" frame   S': the CM frame

(1) how does the CM frame move in S? ⇒ what's $\vec{u}$?

$$\vec{u} = \frac{\vec{P}}{M} = \frac{\sum_i m_i \vec{v}_i}{M}$$ where $M \equiv \sum_j m_j$ is the total mass of the system.

$\vec{u}$: weigh velocities with respect to masses

Momentem in the CM frame: $\vec{P}'$

$$\vec{P}' = \sum m_i \vec{v}_i' = \sum m_i\left(\vec{v}_i - \frac{\vec{P}}{M}\right) = \vec{P} - \vec{P} = \vec{0}$$

why we define $\vec{u}$ like this?
since this gives us $\vec{P}' = \vec{0}$

(2) what is the position of the center of mass (in frame S): What's $\vec{R}_{cm}$?

$$\vec{u} = \frac{\sum m_i \vec{v}_i}{M} \Rightarrow \vec{R}_{cm} = \frac{\sum m_i \vec{r}_i}{M}$$

weigh positions with respect to masses

(3) where to put the ORIGIN of the coordinate of the CM frame?

Answer: just at $\vec{R}_{cm}$

since $\dfrac{d\vec{R}_{cm}}{dt} = \dfrac{\sum m_i \vec{v}_i}{M} = \vec{u}$ (in S ⇒ $\vec{0}$ in S'), which means it doesn't move in the CM frame.

③ König's theorem: About Kinetic Energy ...

In CM Frame: $KE_{cm} = \frac{1}{2}\sum_i m_i |\vec{v}_i'|^2$

In Inertial Frame: $KE_s = \frac{1}{2}\sum m_i |\vec{v}_i|^2 = \frac{1}{2}\sum_i m_i |\vec{v}_i' + \vec{u}|^2$

$$= \frac{1}{2}\sum m_i(\vec{v}_i'\cdot\vec{v}_i' + 2\vec{v}_i'\cdot\vec{u} + \vec{u}\cdot\vec{u})$$

$$= \frac{1}{2}\sum_i m_i |\vec{v}_i'|^2 + \vec{u}\cdot\underbrace{\left(\sum m_i \vec{v}_i'\right)}_{\vec{0}\text{ by definition!}} + \frac{1}{2}|\vec{u}|^2 \sum_i m_i$$

$$= \underbrace{KE_{cm}}_{\text{kinetic energy in the CM frame}} + \underbrace{\tfrac{1}{2}Mu^2}_{\text{kinetic energy of "the center of mass" (always trivial)}}$$

■ Significance of CM frame

(1) physical processes more symmetrical ...

(2) results more transparent ...

(3) important in rigid systems (which you will learn later?) ...

IMPORTANT CONCLUSION:

conservation of energy in the Inertial frame

$\Updownarrow$ equivalent

——————— in the CM frame

Yes! ^^

Reduced Mass

Background: why we are introducing REDUCED MASS?

situation: you are dealing with 2 objects interaction by central force

$m_1 : \vec{r}_1$

$m_2 : \vec{r}_2$

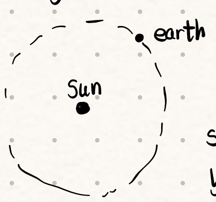
$F(r)$, $V(r)$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$

$r = |\vec{r}|$

$\vec{r}_1$, $\vec{r}_2$, r in an equation $\Rightarrow$ complicated!!! $\Rightarrow$ what if we get rid of $\vec{r}_1$ and $\vec{r}_2$, substitude them by $\vec{r}$?

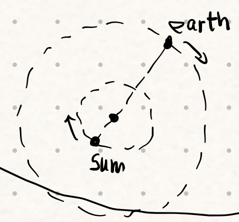
suppose you are solving "the earth orbiting the sun"



since $m_{sun} \gg m_{earth}$

you solve it as if the sun is fixed!

What if m_{sun} not extremely larger than m_{earth}?



as said before, it's complicated

but with reduced mass.

we can convert solving this to ...

equivalent!

Kinetic Energy: $T = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2$

"$\equiv$" means "define"

we define: center of mass: $\vec{R} \equiv \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$

relative position: $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$

$\rightarrow \begin{cases} \vec{r}_1 = \vec{R} + \frac{m_2}{M} \vec{r} \\ \vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{r} \end{cases}$

Then $T = \frac{1}{2} m_1 (\dot{\vec{R}} + \frac{m_2}{M} \dot{\vec{r}})^2 + \frac{1}{2} m_2 (\dot{\vec{R}} - \frac{m_1}{M} \dot{\vec{r}})^2$ $\quad (M = m_1 + m_2)$

$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) \dot{\vec{r}}^2$

we define Reduced Mass: $\mu \equiv \frac{m_1 m_2}{m_1 + m_2} \Leftrightarrow \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

As discussed before — ignore! ^^

$T = \underbrace{\frac{1}{2} M \dot{\vec{R}}^2}_{\frac{1}{2} M \vec{u}^2} + \underbrace{\frac{1}{2} \mu \dot{\vec{r}}^2}_{KE_{CM}}$

trivial

ignore it!

in an "energy" point of view

in CM frame, the kin energy is

equivalent of a particle of mass μ

orbiting a fixed point

what we're concerned about:

a mass of μ with a "motion parameter" of $\vec{r}$

which is also the argument in $F(r)$, $V(r)$ ...