$M = U \quad \Sigma \quad V^*$

- M: $m \times n$, arbitrary matrix
- U: $m \times m$, unitary matrix, $U^*U = I$, rotation/reflection
- Σ: $m \times n$, rescaling
- V^*: $n \times n$, unitary matrix, $V^*V = I$, rotation/reflection

Ex. 1

$$M = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

$$M^TM = \begin{pmatrix} 9 & 3 \\ 3 & 5 \end{pmatrix} \qquad MM^T = \begin{pmatrix} 10 & 3 \\ 3 & 5 \end{pmatrix}$$

$$\det(M^TM - \lambda I) = 0 \Rightarrow \begin{cases} \lambda = 10 & \vec{v}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \vec{u}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ \lambda = 4 & \vec{v}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \vec{u}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3 \end{pmatrix} \end{cases}$$

$$\Sigma = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 2 \end{pmatrix} \qquad V^T = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}^T \qquad U = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{pmatrix}$$

singular values:

$\sigma_1 = \sqrt{10}$, $\sigma_2 = 2$

Ex. 2

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} -0.2298 & 0.8835 & 0.4082 \\ -0.5247 & 0.2408 & -0.8165 \\ -0.8196 & -0.4019 & 0.4082 \end{pmatrix} \begin{pmatrix} 9.5255 & 0 \\ 0 & 0.5143 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -0.6196 & -0.7849 \\ -0.7849 & 0.6196 \end{pmatrix}$$

singular values:

$\sigma_1 = 9.5255$, $\sigma_2 = 0.5143$

Ex. 3

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -0.4046 & -0.9145 \\ -0.9145 & 0.4046 \end{pmatrix} \begin{pmatrix} 9.5080 & 0 & 0 \\ 0 & 0.7729 & 0 \end{pmatrix} \begin{pmatrix} -0.4287 & -0.8059 & -0.4082 \\ -0.5663 & 0.1124 & -0.8165 \\ -0.7039 & -0.5811 & 0.4082 \end{pmatrix}$$

singular values:

$\sigma_1 = 9.5080$, $\sigma_2 = 0.7729$, $\sigma_3 = 0$