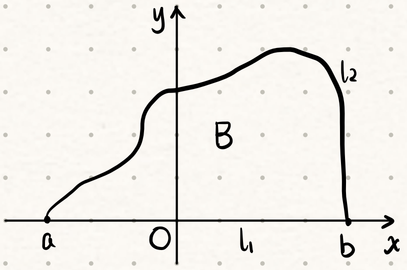



$\int_a^b f(x)dx \Rightarrow \oint_l f(z) = \int_{l_1} f(x)dx + \underbrace{\int_{l_2} f(z)dz}_{\text{往往证明为零}}$

类型一 $\int_0^{2\pi} R(\cos x, \sin x)\,dx$

$z=e^{ix} \Rightarrow \cos x = \frac{1}{2}(z+z^{-1}),\ \sin x = \frac{1}{2i}(z-z^{-1}),\ dx = \frac{1}{iz}dz$

$I = \oint_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$

例1 $I = \int_0^{2\pi}\frac{dx}{1+\varepsilon\cos x}$ $(0<\varepsilon<1)$

$= \oint_{|z|=1}\frac{\frac{dz}{iz}}{1+\varepsilon\frac{z+z^{-1}}{2}}$

$= \frac{2}{i}\oint_{|z|=1}\frac{dz}{\varepsilon z^2+2z+\varepsilon}$

单极点 $z_0 = \frac{-1-\sqrt{1-\varepsilon^2}}{\varepsilon}$ 在单位圆外

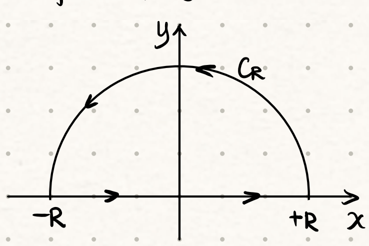
单极点 $z_0 = \frac{-1-\sqrt{1-\varepsilon^2}}{\varepsilon}$ 在单位圆内

$\mathrm{Res}\, f\left(\frac{-1+\sqrt{1-\varepsilon^2}}{\varepsilon}\right) = \lim_{z\to z_0}\frac{1}{(\varepsilon z^2+2z+\varepsilon)'}$

$= \lim_{z\to z_0}\frac{1}{2\varepsilon z+2} = \frac{1}{2\sqrt{1-\varepsilon^2}}$

$I = \frac{2}{i}\, 2\pi i\,\frac{1}{2\sqrt{1-\varepsilon^2}} = \frac{2\pi}{\sqrt{1-\varepsilon^2}}$

类型二 $\int_{-\infty}^{\infty} f(x)dx$；$f(z)$ 在实轴上无奇点，在上半平面除有限个奇点外是解析的；当 $z$ 在上半面及实轴上 $\to\infty$ 时，$zf(z)$ 一致地 $\to 0$




$\oint_l f(z)dz = \int_{-R}^{R} f(x)dx + \underbrace{\int_{C_R} f(z)dz}_{0}$ 则

$\int_{-\infty}^{\infty} f(x)dx = 2\pi i\{f(z)$在上半平面所有奇点的留数之和$\}$

例3 $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$

$f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$

$\mathrm{Res}\, f(+i) = \lim_{z\to i}[(z-i)f(z)] = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}$

$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = 2\pi i\left\{\frac{1}{2i}\right\} = \pi$

类型三 $\int_0^{\infty}\underset{\text{偶}}{F(x)}\cos mx\,dx$，$\int_0^{\infty}\underset{\text{奇}}{G(x)}\sin mx\,dx$ $F(z)$和$G(z)$在实轴上无奇点，

在上半平面除有限个奇点外是解析的；

当 $z$ 在上半平面或实轴上 $\to\infty$ 时，$F(z)$及$G(z)$一致地 $\to 0$.

$\int_0^{\infty} F(x)\cos mx\,dx = \frac{1}{2}\int_{-\infty}^{\infty} F(x)e^{imx}dx$

$\int_0^{\infty} G(x)\sin mx\,dx = \frac{1}{2i}\int_{-\infty}^{\infty} G(x)e^{imx}dx$

约当引理：$\lim_{R\to\infty}\int_{C_R} F(z)e^{imz}dz = 0$

$\lim_{R\to\infty}\int_{C_R} G(z)e^{imz}dz = 0$

于是

$\int_0^{\infty} F(x)\cos mx\,dx = \pi i\{F(z)e^{imx}$在上半平面所有奇点的留数之和$\}$

$\int_0^{\infty} G(x)\sin mx\,dx = \pi\{G(z)e^{imx}$在上半平面所有奇点的留数之和$\}$

例7 $\int_0^{\infty}\frac{x\sin mx}{(x^2+a^2)^2}dx$

$G(z)e^{imz} = \frac{z}{(z^2+a^2)^2}e^{imz}$ 上半平面：二阶极点 $+ai$

$\mathrm{Res}\cdots = \lim_{z\to ai}\frac{1}{1!}\frac{d}{dz}\left[(z-ai)^2\frac{z}{(z^2+a^2)^2}e^{imz}\right]$

$= \lim_{z\to ai}\frac{d}{dz}\left[\frac{z}{(z+ai)^2}e^{imz}\right]$

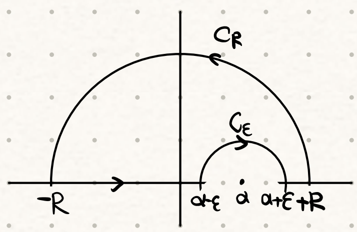
$= \lim_{z\to ai}\left[\frac{1}{(z+ai)^2}e^{imz} + \frac{z}{(z+ai)^2}ime^{imz} - 2\frac{z}{(z+ai)^3}e^{imz}\right]$

$= -\frac{1}{4a^2}e^{-ma} + \frac{ma}{4a^2}e^{-ma} + \frac{1}{4a^2}e^{-ma}$

$= \frac{m}{4a}e^{-ma}$

$\int_0^{\infty}\frac{x\sin mx}{(x^2+a^2)^2}dx = \pi\left(\frac{m}{4a}e^{-ma}\right) = \frac{m\pi}{4a}e^{-ma}$

实轴上有单极点的情形 $\int_{-\infty}^{\infty} f(x)dx$




$\int_{-\infty}^{\infty} f(x)dx = 2\pi i\sum_{\text{上半面}}\mathrm{Res}\, f(z) + \pi i\sum_{\text{实轴上}}\mathrm{Res}\, f(z)$

例8 $\int_0^{\infty}\frac{\sin x}{x}dx$

$= \frac{1}{2i}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx$

$= \frac{\pi}{2}\left\{\frac{e^{iz}}{z}\text{在单极点}z=0\text{的留数}\right\} = \frac{\pi}{2}.$