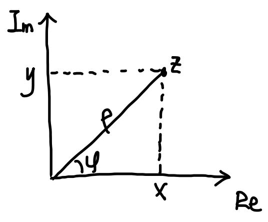
复变函数

2020年6月18日 星期四　08:57

复数



$z = x + iy$

$z = \rho(\cos\varphi + i\sin\varphi) = \rho e^{i\varphi}$

模 $\rho = |z|$

幅角 $\varphi = \mathrm{Arg}\, z$　主值 $0 \le \arg z < 2\pi$　$\varphi = \arg z + 2\pi$

复共轭 $z^* = x - iy = \rho e^{-i\varphi}$

幂：$z^n = \rho^n e^{in\varphi}$　$z^{\frac{1}{n}} = \rho^{\frac{1}{n}} e^{\frac{i(\varphi + 2k\pi)}{n}}$

复变函数 (例子)

$f(z) = a_0 + a_1 z + a_2 z + \cdots + a_n z$

$f(z) = \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

$f(z) = \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$f(z) = \mathrm{sh}\, z = \frac{1}{2}(e^{z} - e^{-z})$

$f(z) = \mathrm{ch}\, z = \frac{1}{2}(e^{z} + e^{-z})$

多值

$f(z) = \ln z = \ln(|z| e^{i\mathrm{Arg}\, z}) = \ln|z| + i\mathrm{Arg}\, z$

$f(z) = u(x,y) + iv(x,y)$

可写为 $f(z) = \ldots$

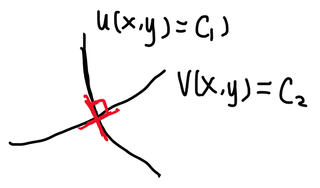
$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$ or $\begin{cases} \frac{\partial u}{\partial \rho} = \frac{1}{\rho}\frac{\partial v}{\partial \varphi} \\ \frac{\partial v}{\partial \rho} = -\frac{1}{\rho}\frac{\partial u}{\partial \varphi} \end{cases}$

＋偏导数连续

解析函数 (可导)

对 u,法 $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$

切 $(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}) = -\nabla u$



$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) u = 0$

$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) v = 0$

虚实知一可互求

1. 特殊路径积分法
2. 凑全微分显示法
3. 不定积分法

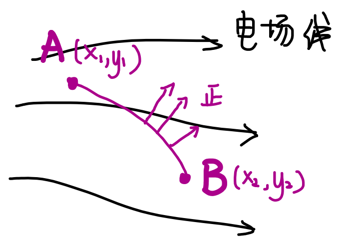
N2,2

平面标量场 (在空间某一方向上均匀)

对平面静电场

$u(x,y)$ 为电势　　$V(x,y)$ 称通量函数

$u(x,y) = C_1$ 为等势线　　$V(x,y) = C_2$ 为电场线



场强 $\vec{E} = -\nabla u = (-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y})$

电荷密度 $\vec{E}\cdot\vec{n} = \frac{\partial u}{\partial x}\frac{dy}{ds} - \frac{\partial u}{\partial y}\frac{dx}{ds}$

其中 $ds = \sqrt{(dx)^2 + (dy)^2}$

$N = \int_A^B \vec{E}\cdot d\vec{s} = \int_A^B \vec{E}\cdot\vec{n}\, ds$

$= \int_A^B dv = v(x_2, y_2) - V(x_1, y_1)$

N3,2
H1,1
H1,2

多值函数

$\ln z = \ln|z| + i\mathrm{Arg}\, z = r + i\arg z + i2n\pi \quad (n = 0, \pm1, \pm2 \cdots)$

支点　割线　Riemann 面

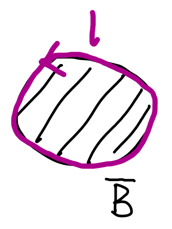
见 N4,1

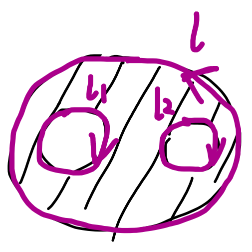
别忘了 $z = \infty$

通常 $f(z) = z^{\frac{1}{n}}$ 的支点是 n-1 阶支点

复变函数的积分

柯西定理





$\oint_l f(z)\, dz = 0$　　$\oint_l f(z)\, dz + \sum \oint_{l_i} f(z)\, dz = 0$

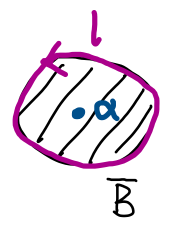
⇓

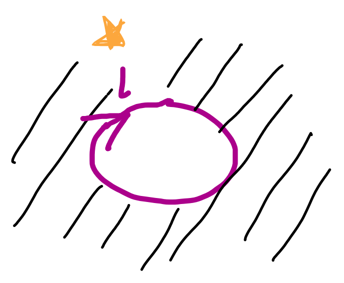
$I = \oint_l (z-\alpha)^n dz$ (其中 n 为整数)

$= \begin{cases} 2\pi i & (n = -1 \text{ 且 } l \text{ 包含 } \alpha) \\ 0 & (\text{其它}) \end{cases}$

⇓

柯西公式





$f(\alpha) = \frac{1}{2\pi i}\oint_l \frac{f(z)}{z-\alpha} dz$

$f(z) = \frac{1}{2\pi i}\oint_l \frac{f(\zeta)}{\zeta - z} d\zeta$　　$f(z) = \frac{1}{2\pi i}\oint_{l+\sum l_i} \frac{f(\zeta)}{\zeta - z} d\zeta$　　$f(z) = \frac{1}{2\pi i}\oint_l \frac{f(\zeta)}{\zeta - z} dz + f(\infty)$

推论：解析函数可求导无数多次且都解析

$f(z) = \frac{1}{2\pi i}\oint_l \frac{f(\zeta)}{\zeta - z} d\zeta$　$f^{n}(z) = \frac{n!}{2\pi i}\oint \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$

还有俩推论，我都不考。