拉普拉斯变换的目的：求解微分方程（和傅里叶变换一样）

本质：对于不存在傅里叶变换的函数，乘一个收敛因子使其可作傅里叶变换

研究对象：$f(t),\ t>0$　置 $f(t)=0,\ t<0$

构造 $g(t)=\underbrace{e^{-\sigma t}}_{\text{收敛因子},\ \sigma>0} f(t)$

对 $g(t)$ 施行傅里叶变换 $G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)\,e^{-i\omega t}\,dt$

$$=\frac{1}{2\pi}\int_0^{\infty} f(t)\,e^{-(\sigma-i\omega)t}\,dt$$

记 $p=\sigma+i\omega$，$\bar{f}(p)=2\pi G(\omega)$，则 $\underbrace{\bar{f}(p)}_{f(t)\text{的拉普拉斯变换}}=\int_0^{\infty} f(t)\,e^{-pt}\,dt$

逆变换 $g(t)=\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\sigma+i\omega)e^{i\omega t}\,d\omega$

$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(\sigma+i\omega)e^{(\sigma+i\omega)t}\,d\omega$，由 $\sigma+i\omega=p\Rightarrow d\omega=\frac{1}{i}dp$

$$f(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\bar{f}(p)e^{pt}\,dp$$

记 $\bar{f}(p)=\mathcal{L}[f(t)]$　或 $\bar{f}(p)\doteq f(t)$

$f(t)=\mathcal{L}^{-1}[\bar{f}(p)]$　　$f(t)\doteq \bar{f}(p)$

基本性质

(1) 线性定理　若 $f_1(t)\doteq\bar{f}_1(p)$，$f_2(t)\doteq\bar{f}_2(p)$，

则 $c_1f_1(t)+c_2f_2(t)\doteq c_1\bar{f}_1(p)+c_2\bar{f}_2(p)$

(2) 导数定理　$f'(t)\doteq p\bar{f}(p)-f(0)$

(3) 积分定理　$\int_0^t \varphi(\tau)\,d\tau\doteq\frac{1}{p}\mathcal{L}[\varphi(t)]$

(4) 相似性定理　$f(at)\doteq\frac{1}{a}\bar{f}\left(\frac{p}{a}\right)$

(5) 位移定理　$e^{-\lambda t}f(t)\doteq\bar{f}(p+\lambda)$

(6) 延迟定理　$f(t-t_0)\doteq e^{-pt_0}\bar{f}(p)$

(7) 卷积定理　若 $f_1(t)\doteq\bar{f}_1(p)$，$f_2(t)\doteq\bar{f}_2(p)$，则 $f_1(t)*f_2(t)\doteq\bar{f}_1(p)\,\bar{f}_2(p)$，

其中 $f_1(t)*f_2(t)\equiv\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$

例：傅里叶级数法求解非齐次方程

$$\begin{cases} u_{tt}-a^2u_{xx}=A\cos\frac{\pi x}{l}\sin\omega t \\ u_x|_{x=0}=0,\ u_x|_{x=l}=0 \\ u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x),\quad 0<x<l \end{cases}$$

右手不可傅里叶变换！于是做拉普拉斯变换

解：显然 $u(x,t)=\sum_{n=0}^{\infty}T_n(t)\cos\frac{n\pi x}{l}$，求解 $T_n(t)$

$$\sum_{n=0}^{\infty}\left[T_n''+\frac{n^2\pi^2a^2}{l^2}T_n\right]\cos\frac{n\pi x}{l}=\underbrace{A\cos\frac{\pi x}{l}\sin\omega t}_{n=1\text{项}}$$

于是 $T_1''+\frac{\pi^2a^2}{l^2}T_1=A\sin\omega t$，

$T_n''+\frac{n^2\pi^2a^2}{l^2}T_n=0,\ n\neq1$

代入初始条件

$$\sum_{n=0}^{\infty}T_n(0)\cos\frac{n\pi x}{l}=\varphi(x)=\sum_{n=0}^{\infty}\varphi_n\cos\frac{n\pi x}{l}$$

$$\sum_{n=0}^{\infty}T_n'(0)\cos\frac{n\pi x}{l}=\psi(x)=\sum_{n=0}^{\infty}\psi_n\cos\frac{n\pi x}{l}$$

得 $\begin{cases} T_0(0)=\varphi_0=\frac{1}{l}\int_0^l\varphi(\xi)\,d\xi, \\ T_0'(0)=\psi_0=\frac{1}{l}\int_0^l\psi(\xi)\,d\xi \end{cases}$

$\begin{cases} T_n(0)=\varphi_n=\frac{2}{l}\int_0^l\varphi(\xi)\cos\frac{n\pi\xi}{l}\,d\xi \\ T_n'(0)=\psi_n=\frac{2}{l}\int_0^l\psi(\xi)\cos\frac{n\pi\xi}{l}\,d\xi \end{cases}\quad n\neq0$

得 $T_0(t)=\varphi_0+\psi_0t$

这个通过拉普拉斯换解得

见下页分解

$$T_1(t)=\frac{Al}{\pi a}\frac{1}{\omega^2-\pi^2a^2/l^2}\left(\omega\sin\frac{\pi at}{l}-\frac{\pi a}{l}\sin\omega t\right)+\varphi_1\cos\frac{\pi at}{l}+\frac{l}{\pi a}\psi_1\sin\omega t$$

$$T_n(t)=\varphi_n\cos\frac{n\pi at}{l}+\frac{l}{n\pi a}\psi_n\sin\frac{n\pi at}{l}\quad(n\neq0,1)$$

$$T_1'' + \frac{\pi^2 a^2}{l^2} T_1 = A\sin\omega t$$

由 $f''(t) \doteq p^2\bar{f}(p) - pf(0) - f'(0)$

$$p^2\bar{T}_1 - pT_1(0) - T_1'(0) + \frac{\pi^2 a^2}{l^2}\bar{T}_1 = A\frac{\omega^2}{p^2+\omega^2}$$

代入 $T_1(0)=\varphi_1$, $T_1'(0)=\psi_1$,

$$\bar{T}_1 = \boxed{A\frac{1}{p^2+\frac{\pi^2a^2}{l^2}}\frac{\omega}{p^2+\omega^2}} + \frac{p\varphi_1}{p^2+\frac{\pi^2a^2}{l^2}} + \frac{\psi_1}{p^2+\frac{\pi^2a^2}{l^2}}$$

$$\varphi_1\frac{p}{p^2+\frac{\pi^2a^2}{l^2}} \doteq \varphi_1\cos\frac{\pi a t}{l}$$

$$\psi_1\frac{l}{\pi a}\frac{\frac{\pi a}{l}}{p^2+\frac{\pi^2a^2}{l^2}} \doteq \psi_1\frac{l}{\pi a}\sin\frac{\pi a t}{l}$$

$$\frac{Al}{\pi a}\frac{\frac{\pi a}{l}}{p^2+\frac{\pi^2a^2}{l^2}}\frac{\omega}{p^2+\omega^2}$$

$$\frac{\omega}{p^2+\omega^2} \doteq \sin\omega t$$

$$\frac{\frac{\pi a}{l}}{p^2+\frac{\pi^2a^2}{l^2}} \doteq \sin\frac{\pi a t}{l}$$

利用卷积定理

$$\int_0^t \sin\frac{\pi a\tau}{l}\sin(t-\tau)\,d\tau = \frac{1}{\omega^2-\frac{\pi^2a^2}{l^2}}\left(\omega\sin\frac{\pi a t}{l} - \frac{\pi a}{l}\sin\omega t\right)$$

于是 $T_1(t) = \frac{Al}{\pi a}\frac{1}{\omega^2-\pi^2a^2/l^2}\left(\omega\sin\frac{\pi a t}{l} - \frac{\pi a}{l}\sin\omega t\right) + \varphi_1\cos\frac{\pi a t}{l} + \frac{l}{\pi a}\psi_1\sin\omega t$