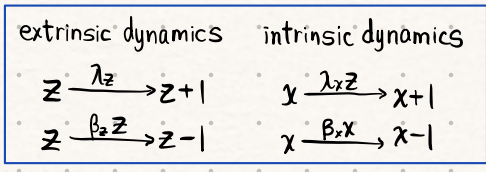


A fluctuating extrinsic variable  $z$  affects the birth rate of the rate of the intrinsic variable  $x$ :



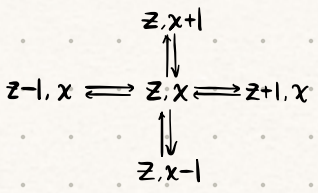
At stationary, the total variance of  $x$  is then

$$\eta_{\text{tot}}^2 = \underbrace{\frac{1}{\langle x \rangle}}_{\eta_{\text{int}}^2} + \underbrace{\frac{1}{\langle z \rangle} \frac{\tau_z}{\tau_x + \tau_z}}_{\eta_{\text{ext}}^2} \quad [2] \quad \text{where } \tau_z = \frac{1}{\beta_z}, \tau_x = \frac{1}{\beta_x}$$

Rewrite the observed variability  $\sigma_x^2$  by conditioning on the state  $z$  of environmental variables

$$\sigma_x^2 = \langle \sigma_{x|z}^2 \rangle + \sigma_{\langle x|z \rangle}^2 \quad (\text{the law of total variance})$$

Calculate  $\sigma_{x|z}^2$  from the master equation



$$\begin{aligned} \frac{dP(z, x)}{dt} = & -\lambda_z P(z, x) + \lambda_z P(z+1, x) \\ & + \beta_z (z+1) P(z+1, x) - \beta_z z P(z, x) \\ & - \lambda_x z P(z, x) + \lambda_x (z-1) P(z, x) \\ & + \beta_x (x+1) P(z, x+1) - \beta_x x P(z, x) \end{aligned}$$

$$\begin{aligned} \frac{d\langle x|z \rangle}{dt} = & \sum_x x \frac{dP(z, x)}{dt} \\ = & -\lambda_z \sum_x x P(z, x) + \lambda_z \sum_x x P(z+1, x) \\ & + \beta_z (z+1) \sum_x x P(z+1, x) - \beta_z z \sum_x x P(z, x) \\ & - \lambda_x z \sum_x x P(z, x) + \lambda_x (z-1) \sum_x x P(z, x) \\ & + \beta_x \sum_x (x+1) x P(z, x) - \beta_x \sum_x x^2 P(z, x) \\ = & -\lambda_z \langle x|z \rangle P(z) + \lambda_z \langle x|z+1 \rangle P(z+1) \\ & + \beta_z (z+1) \langle x|z+1 \rangle P(z+1) - \beta_z z \langle x|z \rangle P(z) \\ & - \lambda_x z \langle x|z \rangle P(z) + \lambda_x (z-1) \langle x|z \rangle P(z) + \lambda_x z P(z) \\ & + \beta_x \langle x^2|z \rangle P(z) - \beta_x \langle x|z \rangle P(z) - \beta_x \langle x|z \rangle P(z) \\ = & -\lambda_z \langle x|z \rangle P(z) + \beta_z z \langle x|z-1 \rangle P(z) \\ & + \lambda_z \langle x|z+1 \rangle P(z) - \beta_z z \langle x|z \rangle P(z) \\ & + \lambda_x z P(z) - \beta_x \langle x|z \rangle P(z) \end{aligned}$$

"index shift" [2]

Stationary:  $\frac{d\langle x|z \rangle}{dt} = 0$

$$\forall z, \lambda_x z = \beta_x \langle x|z \rangle - \beta_z z (\langle x|z+1 \rangle - \langle x|z \rangle) - \lambda_z (\langle x|z+1 \rangle - \langle x|z \rangle)$$

Notation here:  $\langle \langle x|z \rangle \rangle_z = \sum_z \langle x|z \rangle P(z) = A + B\langle z \rangle$   
 the outer bracket  $\langle \langle x|z \rangle^2 \rangle_z = \sum_z \langle x|z \rangle^2 P(z) = A^2 + 2AB\langle z \rangle + B^2\langle z^2 \rangle$   
 indicate averages over all states of  $z$ .

Guess from linearity  $\Rightarrow \langle x|z \rangle = A + Bz$

Solve for A & B  $\rightarrow \sum_z \langle x|z \rangle P(z) = \langle x \rangle = \frac{\lambda_x \lambda_z}{\beta_x \beta_z}$   
 $\Rightarrow A + \frac{\lambda_z}{\beta_z} B = \frac{\lambda_x \lambda_z}{\beta_x \beta_z}$

result

$$\begin{cases} A = \frac{1}{\beta_x + \beta_z} \frac{\lambda_x \lambda_z}{\beta_z} \\ B = \frac{1}{\beta_x + \beta_z} \lambda_x \end{cases}$$

$$\begin{aligned} \sigma_{x|z}^2 = & \langle \langle x|z \rangle^2 \rangle_z - \langle \langle x|z \rangle \rangle_z^2 \\ = & B^2 (\langle z^2 \rangle - \langle z \rangle^2) \quad \text{where } z \text{ is Poisson distributed} \\ = & B^2 \langle z \rangle \end{aligned}$$

$$\begin{aligned} \frac{\sigma_{x|z}^2}{\langle x \rangle^2} = & \frac{\lambda_x^2}{(\beta_x + \beta_z)^2} \frac{\langle z \rangle}{\langle x \rangle^2} = \frac{1}{\langle z \rangle} \frac{\tau_z^2}{(\tau_x + \tau_z)^2} \\ = & \eta_{\text{ext}}^2 \frac{\tau_z}{\tau_x + \tau_z} \end{aligned}$$

The mathematical details of calculating  $\sigma_{x|z}^2$  analytically is shown as above.

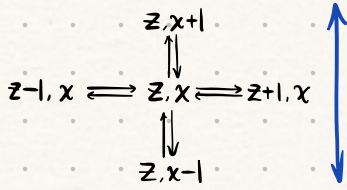
Likewise, I guess we can calculate  $\langle \sigma_{x|z}^2 \rangle$  by  $\sum_x x^2 \frac{dP(x, z)}{dt} = \frac{d\langle x^2|z \rangle}{dt} = 0$ , which gives us  $\langle x^2|z \rangle$  as a function of  $z$ .

$$\begin{aligned} \langle \sigma_{x|z}^2 \rangle = & \langle \langle x^2|z \rangle \rangle_z - \langle \langle x|z \rangle^2 \rangle_z \\ = & \sum_z \langle x^2|z \rangle P(z) - \sum_z \langle x|z \rangle^2 P(z). \end{aligned}$$

but I don't bother to do so.

The result should be  $\frac{\langle \sigma_{x|z}^2 \rangle}{\langle x \rangle^2} = \frac{1}{\langle x \rangle^2} + \frac{1}{\langle z \rangle} \frac{\tau_z}{\tau_x + \tau_z} (1 - \frac{\tau_z}{\tau_x + \tau_z})$ .

## Decomposing Noise by the History of the Environment



$$\frac{d\langle x_t | z_{[0,t]} \rangle}{dt} = \lambda_x z(t) - \beta_x \langle x_t | z_{[0,t]} \rangle$$

Denote  $\langle x_t | z_{[0,t]} \rangle$  as  $\bar{x}$ ,  $\langle x_t^2 | z_{[0,t]} \rangle$  as  $\overline{x^2}$

That is,  $\bar{\cdot}$  for the ensemble average, and see below,

$\langle \dots \rangle_{z_{[0,t]}}$  for long time average.

$$\begin{aligned} \langle \overline{x^2} \rangle_{z_{[0,t]}} - \langle \overline{x^2} \rangle_{z_{[0,t]}} &= \langle \overline{x^2} - \bar{x}^2 \rangle_{z_{[0,t]}} \\ &= \langle \langle x_t^2 | z_{[0,t]} \rangle - \langle x_t | z_{[0,t]} \rangle^2 \rangle_{z_{[0,t]}} \\ &= \langle \sigma_{x_t | z_{[0,t]}}^2 \rangle_{z_{[0,t]}} = \langle \sigma \rangle \end{aligned}$$

$$\Rightarrow \frac{\langle \sigma_{x_t | z_{[0,t]}}^2 \rangle_{z_{[0,t]}}}{\langle \sigma \rangle} = \frac{1}{\langle \sigma \rangle} = \eta_{int}^2$$

$$\frac{d\langle \bar{x} \rangle}{dt} = 2\lambda_x z(t) \bar{x} + \lambda_x z(t) - 2\beta_x \overline{x^2} + \beta_x \bar{x} \quad [4]$$

$$\Rightarrow \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d\langle \bar{x} \rangle^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt [2\lambda_x z(t) \bar{x} + \lambda_x z(t) - 2\beta_x \overline{x^2} + \beta_x \bar{x}]$$

$$0 = 2\lambda_x \langle z(t) \bar{x} \rangle_{z_{[0,t]}} + \lambda_x \langle z \rangle - 2\beta_x \langle \overline{x^2} \rangle_{z_{[0,t]}} + \beta_x \langle \bar{x} \rangle_{z_{[0,t]}}$$

annoying term, however,

$$0 = 2\lambda_x \langle z(t) \bar{x} \rangle_{z_{[0,t]}} + \lambda_x \frac{\lambda_z}{\beta_z} - 2\beta_x \langle \overline{x^2} \rangle_{z_{[0,t]}} + \frac{\lambda_x \lambda_z}{\beta_z}$$

$$\langle \overline{x^2} \rangle_{z_{[0,t]}} - \frac{\lambda_x}{\beta_x} \langle z(t) \bar{x} \rangle_{z_{[0,t]}} = \frac{\lambda_x \lambda_z}{\beta_x \beta_z}, \text{ which is just } \langle \sigma \rangle$$

$$\frac{d\bar{x}}{dt} = \lambda_x z(t) - \beta_x \bar{x} \quad [2]$$

$$\Rightarrow \bar{x} \frac{d\bar{x}}{dt} = \lambda_x z(t) \bar{x} - \beta_x \bar{x}^2$$

"make" a total derivative

$$\frac{1}{2} \frac{d\langle \bar{x} \rangle^2}{dt} = \lambda_x z(t) \bar{x} - \beta_x \bar{x}^2$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{1}{2} d\langle \bar{x} \rangle^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt [\lambda_x z(t) \bar{x} - \beta_x \bar{x}^2]$$

$$0 = \lambda_x \langle z(t) \bar{x} \rangle_{z_{[0,t]}} - \beta_x \langle \bar{x}^2 \rangle_{z_{[0,t]}}$$

now we can cancel this annoying term

$$\frac{\lambda_x}{\beta_x} \langle z(t) \bar{x} \rangle_{z_{[0,t]}} = \langle \bar{x}^2 \rangle_{z_{[0,t]}}$$

It seems that we cannot calculate  $\sigma^2(x_t | z_{[0,t]})$  directly because of the  $\langle z(t) \bar{x} \rangle_{z_{[0,t]}}$  term.

## Intrinsic Noise of Burst Production

$$x \xrightarrow{\lambda(z)} x+b \quad (b \in \mathbb{N}^+, b > 1)$$

$$x \xrightarrow{\beta(z)} x-1$$

$$\langle \sigma_{x_t | z_{[0,t]}}^2 \rangle_{z_{[0,t]}} = \langle \overline{x^2} \rangle - \langle \bar{x} \rangle^2$$

$$\frac{d\langle \bar{x} \rangle}{dt} = 2b \lambda(z) \bar{x} + b^2 \lambda(z) - 2\beta(z) \overline{x^2} + \beta(z) \bar{x}$$

$$\Rightarrow 0 = 2b \langle \lambda(z) \bar{x} \rangle_{z_{[0,t]}} + b^2 \langle \lambda(z) \rangle_{z_{[0,t]}}$$

$$- 2 \langle \beta(z) \overline{x^2} \rangle_{z_{[0,t]}} + \langle \beta(z) \bar{x} \rangle_{z_{[0,t]}}$$

$$\frac{d\bar{x}}{dt} = b \lambda(z) - \beta(z) \bar{x}$$

$$\Rightarrow 0 = b \langle \lambda(z) \rangle_{z_{[0,t]}} - \langle \beta(z) \bar{x} \rangle_{z_{[0,t]}}$$

$$\Rightarrow \bar{x} \frac{d\bar{x}}{dt} = b \lambda(z) \bar{x} - \beta(z) \bar{x}^2$$

$$\frac{1}{2} \frac{d\langle \bar{x} \rangle^2}{dt} = b \lambda(z) \bar{x} - \beta(z) \bar{x}^2$$

$$0 = b \langle \lambda(z) \bar{x} \rangle_{z_{[0,t]}} - \langle \beta(z) \bar{x}^2 \rangle_{z_{[0,t]}}$$