A fluctuating extrinsic variable z affects the birth rate of the rate of the intrinsic variable x:

extrinsic dynamics	intrinsic dynamics
$z \xrightarrow{\lambda_z} z+1$	$x \xrightarrow{\lambda_x z} x+1$
$z \xrightarrow{\beta_z z} z-1$	$x \xrightarrow{\beta_x x} x-1$

At stationary, the total variance of x is then

$$\eta^2_{tot} = \underbrace{\frac{1}{\langle x\rangle}}_{\eta^2_{int}} + \underbrace{\frac{1}{\langle z\rangle}\frac{\tau_z}{\tau_x+\tau_z}}_{\eta^2_{ext}} \quad [2] \quad \text{where } \tau_z = \frac{1}{\beta_z},\ \tau_x = \frac{1}{\beta_x}$$

Rewrite the observed variability σ_x^2 by conditioning on the state $\vec{z}$ of environmental variables

$\sigma_x^2 = \langle \sigma^2_{x|\vec{z}}\rangle + \sigma^2_{\langle x|\vec{z}\rangle}$ (the law of total variance)

Calculate $\sigma^2_{\langle x|\vec{z}\rangle}$ from the master equation

$$\begin{array}{ccccc} & & z,x+1 & & \\ & & \upharpoonleft\downharpoonright & & \\ z-1,x & \rightleftharpoons & z,x & \rightleftharpoons & z+1,x \\ & & \upharpoonleft\downharpoonright & & \\ & & z,x-1 & & \end{array}$$

$$\begin{aligned}\frac{dP(z,x)}{dt} = &-\lambda_z P(z,x) + \lambda_z P(z-1,x) \\ &+ \beta_z(z+1)P(z+1,x) - \beta_z z P(z,x) \\ &- \lambda_x z P(z,x) + \lambda_x z P(z,x-1) \\ &+ \beta_x(x+1)P(z,x+1) - \beta_x x P(z,x)\end{aligned}$$

$$\begin{aligned}\frac{d\langle x|z\rangle}{dt} &= \sum_x x \frac{dP(z,x)}{dt} \\ &= -\lambda_z \sum_x x P(x,z) + \lambda_z \sum_x x P(x,z-1) \\ &\quad + \beta_z(z+1)\sum_x x P(x,z+1) - \beta_z z \sum_x x P(x,z) \\ &\quad - \lambda_x z \sum_x x P(z,x) + \lambda_x z \sum_x (x+1) P(z,x) \\ &\quad + \beta_x \sum_x (x-1) x P(z,x) - \beta_x \sum_x x^2 P(z,x) \\ &= -\lambda_z\langle x|z\rangle P(z) + \lambda_z \langle x|z-1\rangle P(z-1) \\ &\quad + \beta_z (z+1)\langle x|z+1\rangle P(z+1) - \beta_z z\langle x|z\rangle P(z) \\ &\quad \cancel{-\lambda_x z\langle x|z\rangle P(z) + \lambda_x z\langle x|z\rangle} P(z) + \lambda_x z P(z) \\ &\quad \cancel{+\beta_x\langle x^2|z\rangle P(z)} - \beta_x\langle x|z\rangle P(z) \cancel{- \beta_x\langle x^2|z\rangle P(z)} \\ &= -\lambda_z\langle x|z\rangle P(z) + \beta_z z\langle x|z-1\rangle P(z) \\ &\quad + \lambda_z\langle x|z+1\rangle P(z) - \beta_z z\langle x|z\rangle P(z) \\ &\quad + \lambda_x z P(z) - \beta_x\langle x|z\rangle P(z)\end{aligned}$$

"index shift" [2]

Stationary: $\frac{d\langle x|z\rangle}{dt} = 0$

$\forall z,\ \lambda_x z = \beta_x\langle x|z\rangle - \beta_z z(\langle x|z-1\rangle - \langle x|z\rangle) - \lambda_z(\langle x|z+1\rangle - \langle x|z\rangle).$

Guess from linearity $\Rightarrow \langle x|z\rangle = A + Bz$

Solve for A & B $\rightarrow \sum_z \langle x|z\rangle P(z) = \langle x\rangle = \frac{\lambda_x\lambda_z}{\beta_x\beta_z}$

$\Rightarrow A + \frac{\lambda_z}{\beta_z}B = \frac{\lambda_x\lambda_z}{\beta_x\beta_z}$

plug in

$\downarrow$

result

$$\begin{cases} A = \frac{1}{\beta_x+\beta_z}\frac{\lambda_x\lambda_z}{\beta_z} \\ B = \frac{1}{\beta_x+\beta_z}\lambda_x \end{cases}$$

Notation here: $\langle\langle x|z\rangle_x\rangle_z = \sum_z \langle x|z\rangle P(z) = A + B\langle z\rangle$

the outer bracket indicate averages over all states of z.

$\langle\langle x|z\rangle_x^2\rangle_z = \sum_z \langle x|z\rangle^2 P(z) = A^2 + 2AB\langle z\rangle + B^2\langle z^2\rangle$

$$\begin{aligned}\sigma^2_{\langle x|\vec{z}\rangle} &= \langle\langle x|z\rangle_x^2\rangle_z - \langle\langle x|z\rangle_x\rangle_z^2 \\ &= B^2(\langle z^2\rangle - \langle z\rangle^2) \quad \text{where } z \text{ is Poisson distributed} \\ &= B^2\langle z\rangle\end{aligned}$$

$$\frac{\sigma^2_{\langle x|z\rangle}}{\langle x\rangle^2} = \frac{\lambda_x^2}{(\beta_x+\beta_z)^2}\frac{\langle z\rangle}{\langle x\rangle^2} = \frac{1}{\langle z\rangle}\frac{\tau_z^2}{(\tau_x+\tau_z)^2}$$

$$= \eta^2_{ext}\frac{\tau_z}{\tau_x+\tau_z}$$

The mathematical details of calculating $\sigma^2_{\langle x|z\rangle}$ analytically is shown as above.

Likewise, I guess we can calculate $\langle\sigma^2_{x|z}\rangle$ by $\sum_x x^2\frac{dP(x,z)}{dt} = \frac{d\langle x^2|z\rangle}{dt} = 0$, which gives us $\langle x^2|z\rangle$ as a function of z.

but I don't bother to do so.

$$\begin{aligned}\langle\sigma^2_{x|z}\rangle &= \langle\langle x^2|z\rangle_x\rangle_z - \langle\langle x|z\rangle_x^2\rangle_z \\ &= \sum_z\langle x^2|z\rangle_x P(z) - \sum_z\langle x|z\rangle_x^2 P(z).\end{aligned}$$

The result should be $\frac{\langle\sigma^2_{x|z}\rangle}{\langle x\rangle^2} = \frac{1}{\langle x\rangle^2} + \frac{1}{\langle z\rangle}\frac{\tau_z}{\tau_x+\tau_z}\left(1 - \frac{\tau_z}{\tau_x+\tau_z}\right).$

Decomposing Noise by the History of the Environment

$$\begin{tikzcd}
 & z,x+1 \arrow[d, shift left] & \\
z-1,x \arrow[r, shift left] & z,x \arrow[l, shift left] \arrow[r, shift left] \arrow[u, shift left] \arrow[d, shift left] & z+1,x \arrow[l, shift left] \\
 & z,x-1 \arrow[u, shift left] &
\end{tikzcd}$$

$$\frac{d\langle x_t|z[0,t)\rangle}{dt} = \lambda_x z(t) - \beta_x\langle x_t|z[0,t)\rangle$$

Denote $\langle x_t|z[0,t)\rangle$ as $\overline{x}$, $\langle x_t^2|z[0,t)\rangle$ as $\overline{x^2}$

That is, $\overline{\cdots}$ for the ensemble average, and see below,

$\langle\cdots\rangle_{z[0,t)}$ for long time average.

$$\langle\overline{x^2}\rangle_{z[0,t)} - \langle(\overline{x})^2\rangle_{z[0,t]} = \langle\overline{x^2} - (\overline{x})^2\rangle_{z[0,t)}$$

$$= \langle\langle x_t^2|z[0,t)\rangle - \langle x_t|z[0,t)\rangle^2\rangle_{z[0,t]}$$

$$= \langle\sigma^2_{x_t|z[0,t)}\rangle_{z[0,t)} = \langle x\rangle$$

$$\Rightarrow \frac{\langle\sigma^2_{x_t|z[0,t)}\rangle_{z[0,t]}}{\langle x\rangle^2} = \frac{1}{\langle x\rangle} = \eta^2_{int}$$

$$\frac{d(\overline{x})^2}{dt} = 2\lambda_x z(t)\overline{x} + \lambda_x z(t) - 2\beta_x\overline{x^2} + \beta_x\overline{x} \quad [4]$$

$$\Rightarrow \lim_{T\to\infty}\frac{1}{T}\int_0^T d(\overline{x})^2 = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\left[2\lambda_x z(t)\overline{x} + \lambda_x z(t) - 2\beta_x\overline{x^2} + \beta_x\overline{x}\right]$$

$$0 = 2\lambda_x\boxed{\langle z(t)\overline{x}\rangle_{z[0,t]}} + \lambda_x\langle z\rangle - 2\beta_x\langle\overline{x^2}\rangle_{z[0,t]} + \beta_x\langle\overline{x}\rangle_{z[0,t]}$$

annoying term, however,

$$0 = 2\lambda_x\langle z(t)\overline{x}_t\rangle_{z[0,t]} + \lambda_x\frac{\lambda_z}{\beta_z} - 2\beta_x\langle\overline{x^2}\rangle_{z[0,t)} + \frac{\lambda_x\lambda_z}{\beta_z}$$

$$\langle\overline{x^2}\rangle_{z[0,t]} - \frac{\lambda_x}{\beta_x}\langle z(t)\overline{x}_t\rangle_{z[0,t]} = \frac{\lambda_x\lambda_z}{\beta_x\beta_z}$$, which is just $\langle x\rangle$

$$\frac{d\overline{x}}{dt} = \lambda_x z(t) - \beta_x\overline{x} \quad [2]$$

"make" a total derivative

$$\Rightarrow \overline{x}\frac{d\overline{x}}{dt} = \lambda_x z(t)\overline{x} - \beta_x(\overline{x})^2$$

$$\frac{1}{2}\frac{d(\overline{x})^2}{dt} = \lambda_x z(t)\overline{x} - \beta_x(\overline{x})^2$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{1}{2}d(\overline{x})^2 = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\left[\lambda_x z(t)\overline{x} - \beta_x(\overline{x})^2\right]$$

$$0 = \lambda_x\boxed{\langle z(t)\overline{x}\rangle_{z[0,t]}} - \beta_x\langle(\overline{x})^2\rangle_{z[0,t]}$$

now we can cancel this annoying term

$$\frac{\lambda_x}{\beta_x}\langle z(t)\overline{x}\rangle_{z[0,t]} = \langle(\overline{x})^2\rangle_{z[0,t]}$$

It seems that we cannot calculate $\sigma^2\langle x_t|z[0,t)\rangle$ directly because of the $\langle z(t)\overline{x}\rangle_{z[0,t]}$ term.

Intrinsic Noise of Burst Production

$$x \xrightarrow{\lambda(z)} x+b \quad (b\in\mathbb{N}^+,\ b>1)$$

$$x \xrightarrow{\beta(z)x} x-1$$

$$\langle\sigma^2_{x_t|z[0,t)}\rangle_{z[0,t)} = \langle\overline{x^2} - (\overline{x})^2\rangle_{z[0,t)}$$

$$\frac{d(\overline{x})^2}{dt} = 2b\,\lambda(z)\overline{x} + b^2\lambda(z) - 2\beta(z)\overline{x^2} + \beta(z)\overline{x}$$

$$\Rightarrow 0 = 2b\langle\lambda(z)\overline{x}\rangle_{z[0,t]} + b^2\langle\lambda(z)\rangle_{z[0,t]}$$

$$- 2\langle\beta(z)\overline{x^2}\rangle_{z[0,t]} + \langle\beta(z)\overline{x}\rangle_{z[0,t)}$$

$$\frac{d\overline{x}}{dt} = b\lambda(z) - \beta(z)\overline{x}$$

$$\Rightarrow 0 = b\langle\lambda(z)\rangle_{z[0,t)} - \langle\beta(z)\overline{x}\rangle_{z[0,t)}$$

$$\Rightarrow \overline{x}\frac{d\overline{x}}{dt} = b\lambda(z)\overline{x} - \beta(z)(\overline{x})^2$$

$$\frac{1}{2}\frac{d(\overline{x})^2}{dt} = b\lambda(z)\overline{x} - \beta(z)(\overline{x})^2$$

$$0 = b\langle\lambda(z)\overline{x}\rangle_{z[0,t)} - \langle\beta(z)(\overline{x})^2\rangle_{z[0,t]}$$