$C_{ij} = \langle \Delta x_i \Delta x_j | \vec{z}[0,t] \rangle_e$

$$\boxed{J_{ij} = \sum_k \beta_{jk} S_{ik}}$$

$T_{ijl} = \langle \Delta x_i \Delta x_j \Delta x_l | \vec{z}[0,t] \rangle_e$

$$\frac{dT_{ijl}}{dt} = \sum_{\vec{x}} \Delta x_i \Delta x_j \Delta x_l \frac{dP(\vec{x},t|\vec{z}[0,t])}{dt}$$

$$- \langle \Delta x_i \Delta x_j | \vec{z}[0,t] \rangle_e \frac{d\bar{x}_l}{dt} - \langle \Delta x_j \Delta x_l | \vec{z}[0,t] \rangle_e \frac{d\bar{x}_i}{dt} - \langle \Delta x_l \Delta x_i | \vec{z}[0,t] \rangle_e \frac{d\bar{x}_j}{dt}$$

(1') (2') (3')

$$r_k(\vec{x}, \vec{z}(t)) = \alpha_k(\vec{z}(t)) + \sum_p \beta_{kp} x_p$$

$$\frac{dP(\vec{x},t|\vec{z}[0,t])}{dt} = -\sum_k r_k(\vec{x}, \vec{z}(t)) P(\vec{x}|\vec{z}[0,t]) + \sum_k r_k(\vec{x} - \vec{S}_k, \vec{z}(t)) P(\vec{x} - \vec{S}_k | \vec{z}[0,t])$$

"index shift" only shifts the indexes.. $\Delta x_{i,t} = \underbrace{x_{i,t}}_{\text{index}} - \bar{x}_i(t)$

$$\sum_{\vec{x}} \Delta x_i \Delta x_j \Delta x_l \frac{dP(\vec{x},t|\vec{z}[0,t])}{dt} = -\sum_{\vec{x}} \Delta x_i \Delta x_j \Delta x_l \sum_k r_k(\vec{x}, \vec{z}(t)) P(\vec{x}|\vec{z}[0,t])$$

$$+ \sum_{\vec{x}} \sum_k (\Delta x_i + S_{ki})(\Delta x_j + S_{kj})(\Delta x_l + S_{kl}) r_k(\vec{x}, \vec{z}(t)) P(\vec{x}|\vec{z}[0,t])$$

(1) (2) (3)

$$= \sum_{\vec{x}} \sum_k \big(\Delta x_i \Delta x_j S_{kl} + \Delta x_j \Delta x_l S_{ki} + \Delta x_l \Delta x_i S_{kj}$$

$$+ \Delta x_i S_{kj} S_{kl} + \Delta x_j S_{kl} S_{ki} + \Delta x_l S_{ki} S_{kj} + S_{ki} S_{kj} S_{kl} \big) r_k(\vec{x}, \vec{z}(t)) P(\vec{x}|\vec{z}[0,t])$$

(4) (5) (6) (7)

(7): $\sum_{\vec{x}} \sum_k S_{ki} S_{kj} S_{kl} r_k(\vec{x}, \vec{z}(t)) P(\vec{x}|\vec{z}[0,t]) = \sum_k S_{ki} S_{kj} S_{kl} \langle r_k(\vec{x}, \vec{z}(t)) \rangle_e$

(1)-(3): $\sum_{\vec{x}} \sum_k \Delta x_i \Delta x_j S_{kl} r_k(\vec{x}, \vec{z}(t)) P(\vec{x}|\vec{z}[0,t])$

no ensemble average, throw inside

$$- \Big(\sum_{\vec{x}} \Delta x_i \Delta x_j P(\vec{x}|\vec{z}[0,t]) \Big) \Big(\sum_k S_{kl} r_k(\vec{x}, \vec{z}(t)) \Big)$$

$$= \sum_{\vec{x}} \sum_k \Delta x_i \Delta x_j S_{kl} \big(\alpha_k(\vec{z}) + \sum_p \beta_{kp} x_p \big) P(\vec{x}|\vec{z}[0,t])$$

$\sum_p J_{p,l} \Delta x_p$

$$- \sum_{\vec{x}} \sum_k \Delta x_i \Delta x_j S_{kl} \big(\alpha_k(\vec{z}) + \sum_p \beta_{kp} \bar{x}_p \big) P(\vec{x}|\vec{z}[0,t])$$

$$= \sum_{\vec{x}} \sum_k \Delta x_i \Delta x_j S_{kl} \sum_p \beta_{kp} \Delta x_p \; P(\vec{x}|\vec{z}[0,t])$$

$$= \sum_{\vec{x}} \Delta x_i \Delta x_j \sum_p \Delta x_p \Big(\sum_k \beta_{kp} S_{kl} \Big) P(\vec{x}|\vec{z}[0,t])$$

$$= \sum_{\vec{x}} \Delta x_i \Delta x_j \sum_p \Delta x_p J_{lp} \; P(\vec{x}|\vec{z}[0,t])$$

$$= \sum_p J_{lp} \langle \Delta x_p \Delta x_i \Delta x_j | \vec{z}[0,t] \rangle_e$$

$$= \sum_p J_{lp} T_{ijp}$$

$$= (\hat{J} \cdot T)_{ijl}$$

(4): $\sum_{\vec{x}} \sum_k \Delta x_i S_{kj} S_{kl} r_k(\vec{x}|\vec{z}(t)) P(\vec{x}|\vec{z}[0,t])$

$$= \sum_{\vec{x}} \sum_k \Delta x_i S_{kj} S_{kl} \big(\alpha_k(\vec{z}) + \sum_p \beta_{kp} x_p \big) P(\vec{x}|\vec{z}[0,t]).$$

$$= \sum_{\vec{x}} \sum_k \Delta x_i S_{kj} S_{kl} \big(\alpha_k(\vec{z}) + \sum_p \beta_{kp} x_p \big) P(\vec{x}|\vec{z}[0,t])$$

$$- \Big(\underbrace{\sum_{\vec{x}} \Delta x_i P(\vec{x}|\vec{z}[0,t])}_{\| \; 0} \Big) \Big(\sum_k S_{kj} S_{kl} \big(\alpha_k(\vec{z}) + \sum_p \beta_{kp} \bar{x}_p \big) \Big)$$

$$= \sum_{\vec{x}} \sum_k \Delta x_i S_{kj} S_{kl} \sum_p \beta_{kp} \Delta x_p \; P(\vec{x}|\vec{z}[0,t])$$

$$= \sum_k \sum_p \beta_{kp} S_{kj} S_{kl} \Delta x_i \Delta x_p \; P(\vec{x}|\vec{z}[0,t])$$

$$= \sum_k \sum_p \beta_{kp} S_{kj} S_{kl} C_{ip}.$$