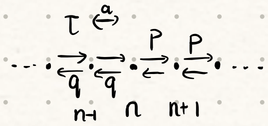
Problem 3. Simple random walk with bias

3.1



$$\begin{cases} n = n_\to - n_\leftarrow \\ N = n_\to + n_\leftarrow \end{cases} \Rightarrow n = 2n_\to - N \qquad p+q=1$$

$$P(n,N) = \frac{N!}{n_\to!\,n_\leftarrow!}\,p^{n_\to}q^{n_\leftarrow} \quad (\text{where } t = N\tau)$$

$$= \frac{N!}{n_\to!(N-n_\to)!}\,p^{n_\to}q^{N-n_\to}$$

$$= \frac{N!}{(\frac{N+n}{2})!\,(\frac{N-n}{2})!}\,p^{\frac{N+n}{2}}(1-p)^{\frac{N-n}{2}}$$

$$\langle n\rangle = \langle 2n_\to - N\rangle = 2\langle n_\to\rangle - N$$

$$\langle n_\to\rangle = \sum_{n_\to=0}^{N} n_\to \frac{N!}{n_\to!(N-n_\to)!}\,p^{n_\to}q^{N-n_\to}$$

$$= \frac{\partial}{\partial\mu}\sum_{n_\to=0}^{N}\frac{N!}{n_\to!(N-n_\to)!}\,p^{n_\to}q^{N-n_\to}e^{\mu n_\to}\Big|_{\mu\to0}$$

$$= q^N\frac{\partial}{\partial\mu}\sum_{n_\to=0}^{N}\frac{N!}{n_\to!(N-n_\to)!}\left(\frac{p}{q}e^\mu\right)^{n_\to}\Big|_{\mu\to0}$$

$$= q^N\frac{\partial}{\partial\mu}\left(1+\frac{p}{q}e^\mu\right)^N\Big|_{\mu\to0}$$

$$= q^N\,N\frac{p}{q}e^\mu\left(1+\frac{p}{q}e^\mu\right)^{N-1}\Big|_{\mu\to0}$$

$$= q^N N\frac{p}{q}\left(1+\frac{p}{q}\right)^{N-1}$$

$$= (1-p)^{N-1}\,Np\left(1+\frac{p}{1-p}\right)^{N-1}$$

$$= Np$$

$$\langle n\rangle = 2\langle n_\to\rangle - N$$

$$= 2Np - N = N(2p-1)$$

$$\text{var}\langle n\rangle = \langle (n-\langle n\rangle)^2\rangle$$

$$= \langle (2n_\to - N - \langle 2n_\to - N\rangle)^2\rangle$$

$$= 4\langle n_\to^2\rangle - 4\langle n_\to\rangle^2$$

$$\langle n_\to^2\rangle = \sum_{n_\to=0}^{N} n_\to^2\frac{N!}{n_\to!(N-n_\to)!}\,p^{n_\to}q^{N-n_\to}$$

$$= \frac{\partial^2}{\partial\mu^2}\sum_{n_\to=0}^{N}\frac{N!}{n_\to!(N-n_\to)!}\,p^{n_\to}q^{N-n_\to}e^{\mu n_\to}\Big|_{\mu\to0}$$

$$= q^N\frac{\partial^2}{\partial\mu^2}\sum_{n_\to=0}^{N}\frac{N!}{n_\to!(N-n_\to)!}\left(\frac{p}{q}e^\mu\right)^{n_\to}\Big|_{\mu\to0}$$

$$= q^N\frac{\partial^2}{\partial\mu^2}\left(1+\frac{p}{q}e^\mu\right)^N\Big|_{\mu\to0}$$

$$= q^N\frac{\partial}{\partial\mu}\,N\frac{p}{q}e^\mu\left(1+\frac{p}{q}e^\mu\right)^{N-1}\Big|_{\mu\to0}$$

$$= q^N N\frac{p}{q}e^\mu\left(1+\frac{p}{q}\right)^{N-2}\left(\frac{p}{q}e^\mu N+1\right)\Big|_{\mu\to0}$$

$$= (1-p)^{N-1}Np\left(1+\frac{p}{1-p}\right)^{N-2}\left(\frac{p}{1-p}N+1\right)$$

$$= Np(pN+1-p)$$

$$\text{var}(n) = 4(\langle n_\to^2\rangle - \langle n_\to\rangle^2)$$

$$= 4[Np(Np+1-p) - (Np)^2]$$

$$= 4Np(1-p)$$

3.2

$$\ln P(n,N) = \ln N! - \ln\left[\left(\tfrac{N+n}{2}\right)!\right] - \ln\left[\left(\tfrac{N-n}{2}\right)!\right] + \frac{N+n}{2}\ln p + \frac{N-n}{2}\ln(1-p)$$

Stirling Approximation: $\ln(N!) \simeq N(\ln N - 1) + \frac12\ln(2\pi N)$

$$\ln P(n,N) = N\ln N - N + \frac12\ln(2\pi N)$$
$$- \frac{N+n}{2}\ln\frac{N+n}{2} + \frac{N+n}{2} - \frac12\ln[\pi(N+n)]$$
$$- \frac{N-n}{2}\ln\frac{N-n}{2} + \frac{N-n}{2} - \frac12\ln[\pi(N-n)]$$
$$+ \frac{N+n}{2}\ln p + \frac{N-n}{2}\ln(1-p)$$

$n = 2Np - N + \delta n \Rightarrow N+n = 2Np + \delta n$

$N-n = 2N(1-p) - \delta n$

$$= N\ln N + \frac12\ln(2\pi N)$$
$$- \frac{N+n}{2}\ln\left[Np\left(1+\frac{\delta n}{2Np}\right)\right] - \frac12\ln\left[2\pi Np\left(1+\frac{\delta n}{2Np}\right)\right]$$
$$- \frac{N-n}{2}\ln\left[N(1-p)\left(1-\frac{\delta n}{2N(1-p)}\right)\right] - \frac12\ln\left[2\pi N(1-p)\left(1-\frac{\delta n}{2N(1-p)}\right)\right]$$
$$+ \frac{N+n}{2}\ln p + \frac{N-n}{2}\ln(1-p)$$

$$= \frac12\ln(2\pi N) - \frac12\ln\left[2\pi Np\left(1+\frac{\delta n}{2Np}\right)\right] - \frac12\ln\left[2\pi N(1-p)\left(1-\frac{\delta n}{2N(1-p)}\right)\right]$$
$$- Np\left(1+\frac{\delta n}{2Np}\right)\ln\left(1+\frac{\delta n}{2Np}\right) - N(1-p)\left(1-\frac{\delta n}{2N(1-p)}\right)\ln\left(1-\frac{\delta n}{2N(1-p)}\right)$$

Taylor expansion: $\ln(1+x) = x - \frac12x^2 + o(x^2)$

$$= -\frac12\ln[2\pi Np(1-p)] - \frac12\ln\left[\left(1+\frac{\delta n}{2Np}\right)\left(1-\frac{\delta n}{2N(1-p)}\right)\right]$$
$$- Np\left(1+\frac{\delta n}{2Np}\right)\left(\frac{\delta n}{2Np} - \frac12\left(\frac{\delta n}{2Np}\right)^2\right) - N(1-p)\left(1-\frac{\delta n}{2N(1-p)}\right)\left(\frac{\delta n}{2N(1-p)} - \frac12\left(\frac{-\delta n}{2N(1-p)}\right)^2\right)$$

$$= -\frac12\ln[2\pi Np(1-p)] - \frac12\ln\left(1+\frac{\delta n}{2N}\frac{1-2p}{p(1-p)} + O\left(\frac{\delta n}{N}\right)^2\right) \quad \text{neglected}$$
$$- Np\left[\frac{\delta n}{2Np} + \frac12\left(\frac{\delta n}{2Np}\right)^2\right] - N(1-p)\left[\frac{-\delta n}{2N(1-p)} + \frac12\left(\frac{-\delta n}{2N(1-p)}\right)^2\right] + O\frac{(\delta n)^3}{N^2}$$

$$= -\frac12\ln[2\pi Np(1-p)] - \frac12\frac{\delta n}{2n}\frac{1-2p}{p(1-p)} - \frac{(\delta n)^2}{8Np} - \frac{(\delta n)^2}{8N(1-p)} + O\left(\frac{\delta n}{N}\right)^2$$

to make it more compact (like Gaussian distribution)

when δn is relatively big, $\delta n \ll (\delta n)^2$. $e^{-\frac12\frac{\delta n}{2n}\frac{1-2p}{p(1-p)}}$ doesn't diverge

since the range of n is finite.

$$\simeq -\frac12\ln[2\pi Np(1-p)] - \frac{(\delta n)^2}{8Np(1-p)}$$

$$= -\frac12\ln\left[\frac{\pi}{2}\text{var}(n)\right] - \frac{(n-\langle N\rangle)^2}{2\text{var}(n)}$$

$$P(n,N) = \frac{2}{\sqrt{2\pi\,\text{var}(n)}}\exp\left[-\frac{(n-\langle N\rangle)^2}{2\,\text{var}(n)}\right]$$

for continuum limit $\begin{cases} x = na \\ t = N\tau \end{cases}$

$$P(x,t) = \frac{1}{2a}P(n,N)$$

$$= \frac{1}{\sqrt{4\pi Dt}}\exp\left[-\frac{(x-vt)^2}{4Dt}\right]$$

where $D = \frac{a^2}{2\tau}[4p(1-p)]$, $v = \frac{a}{\tau}(2p-1)$

$$\langle x\rangle = \int_{-\infty}^{\infty} xP(x,t)\,dt = vt = \frac{a}{\tau}(2p-1)t$$

$$\langle (x-\langle x\rangle)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 \qquad \langle x^2\rangle = \int_{-\infty}^{\infty} x^2P(x,t)\,dt$$

$$= \frac{1}{\sqrt{4\pi Dt}}\sqrt{4\pi Dt}\,(2\cdot Dt + v^2t^2) - v^2t^2$$

$$= 2Dt = 4Np(1-p)a^2$$

↑ integral by Wolfram Alpha

Yes. it does a gree