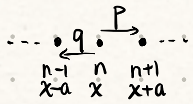
Problem 4: Fokker-Planck approximation

4.1



Master equation:

$$P(x,t+\tau) = p\,P(x-a,t) + (1-p)\,P(x+a,t)$$

when τ is small, $P(x,t+\tau) = P(x,t) + \tau\frac{\partial P}{\partial t}$

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{\tau}\left[p\,P(x-a,t) + (1-p)P(x+a,t) - P(x,t)\right]$$

Fokker-Plank Approximation

when a is small, $P(x\pm a,t) \simeq P(x,t) \pm a\frac{\partial}{\partial x}P(x,t) + \frac{1}{2}a^2\frac{\partial^2}{\partial x^2}P(x,t)$

$$\begin{aligned}\tau\frac{\partial P(x,t)}{\partial t} &= p\,P(x,t) - pa\frac{\partial}{\partial x}P(x,t) + \frac{1}{2}pa^2\frac{\partial^2}{\partial x^2}P(x,t)\\ &\quad + (1-p)\,P(x,t) + (1-p)a\frac{\partial}{\partial x}P(x,t) + \frac{1}{2}(1-p)a^2\frac{\partial^2}{\partial x^2}P(x,t)\\ &\quad - P(x,t)\\ &= a(1-2p)\frac{\partial}{\partial x}P(x,t) + \frac{1}{2}a^2\frac{\partial^2}{\partial x^2}P(x,t)\end{aligned}$$

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= \frac{a}{\tau}(1-2p)\frac{\partial}{\partial x}P(x,t) + \frac{a^2}{2\tau}\frac{\partial^2}{\partial x^2}P(x,t)\\ &= -v\frac{\partial}{\partial x}P(x,t) + D\frac{\partial^2}{\partial x^2}P(x,t) \qquad ①\end{aligned}$$

where $v = \frac{a}{\tau}(2p-1)$, $D = \frac{a^2}{2\tau}$

Use Fourier transformation

$$P_f(f,t) = \int_{-\infty}^{\infty} P(x,t)e^{ifx}\,dx \Leftrightarrow P(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_f(f,t)e^{-ifx}\,df$$

Define operator ϕ: $P(x,t) \xrightarrow{\phi} P_f(f,t)$

$$P_f(f,t) = \phi[P(x,t)] = \int_{-\infty}^{\infty} P(x,t)e^{ifx}\,dx$$

Properties: $\phi\left(\frac{\partial}{\partial x}P(x,t)\right) = -if\,\phi(P(x,t))$

then $\phi\left(\frac{\partial^2}{\partial x^2}P(x,t)\right) = -f^2\,\phi(P(x,t))$

From ①, we use operator ϕ

$$\phi\left(\frac{\partial P(x,t)}{\partial t}\right) = \phi\left[-v\frac{\partial}{\partial x}P(x,t) + D\frac{\partial^2}{\partial x^2}P(x,t)\right]$$

note that no "time" operation in $\phi \Rightarrow \phi\left(\frac{\partial P(x,t)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\phi(P(x,t))\right)$

$$\begin{aligned}\frac{\partial}{\partial t}P_f(f,t) &= ivf\,P_f(f,t) - Df^2P_f(f,t)\\ &= (ivf - Df^2)\,P_f(f,t)\end{aligned}$$

thus, $P_f(f,t) = P_f(f,0)\,e^{(ivf-Df^2)t}$

For initial condition

$P(x,0) = \delta(x)$ $\left(\text{so that } \int_{-\infty}^{\infty} P(x,t)\,dt = 1\right)$

$$P_f(f,0) = \int_{-\infty}^{\infty} P(x,0)e^{ifx}\,dx = e^{if0} = 1$$

thus, $P_f(f,t) = e^{(ivf-Df^2)t}$

$$\begin{aligned}P(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(f,t)e^{-ifx}\,df\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-if(x-vt)-Df^2t}\,df\\ &= \frac{1}{2\pi}e^{-\frac{(x-vt)^2}{4Dt}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{Dt}\,f + i\frac{x-vt}{2\sqrt{Dt}}\right)^2}\,df\\ &= \frac{1}{2\pi}e^{-\frac{(x-vt)^2}{4Dt}}\sqrt{\frac{\pi}{Dt}}\\ &= \frac{1}{\sqrt{4\pi Dt}}\,e^{-\frac{(x-vt)^2}{4Dt}}\end{aligned}$$