Discrete version

$$\begin{cases} n = n_\rightarrow - n_\leftarrow \\ N = n_\rightarrow + n_\leftarrow \end{cases}$$

$p+q=1$

$$P(n,N) = \frac{N!}{n_\rightarrow!\, n_\leftarrow!} p^{n_\rightarrow} q^{n_\leftarrow}$$

Gaussian approximation

$$\langle n \rangle = \langle 2n_\rightarrow - N \rangle = 2\langle n_\rightarrow \rangle - N$$

$$\langle n_\rightarrow \rangle = \sum_{n_\rightarrow=0}^{N} n_\rightarrow \frac{N!}{n_\rightarrow!(N-n_\rightarrow)!} p^{n_\rightarrow} q^{N-n_\rightarrow}$$

$$= \frac{\partial}{\partial\mu} \underbrace{\sum_{n_\rightarrow=0}^{N} \frac{N!}{n_\rightarrow!(N-n_\rightarrow)!} p^{n_\rightarrow} q^{N-n_\rightarrow} e^{\mu n_\rightarrow}}_{\text{Generating function}} \Big|_{\mu\to 0}$$

$$= q^N \frac{\partial}{\partial\mu} \sum_{n_\rightarrow=0}^{N} \frac{N!}{n_\rightarrow!(N-n_\rightarrow)!} \left(\frac{p}{q} e^\mu\right)^{n_\rightarrow} \Big|_{\mu\to 0}$$

$$= q^N \frac{\partial}{\partial\mu} \left(1+\frac{p}{q}e^\mu\right)^N \Big|_{\mu\to 0}$$

$$= q^N N \frac{p}{q} e^\mu \left(1+\frac{p}{q}e^\mu\right)^{N-1} \Big|_{\mu\to 0}$$

$$= (1-p)^N N \frac{p}{1-p} \left(1+\frac{p}{1-p}\right)^{N-1}$$

$$= pN$$

$$\langle n \rangle = 2\langle n_\rightarrow \rangle - N = 2pN - N = N(2p-1)$$

For $p=\frac{1}{2}$, $\langle n \rangle = 0$

$$\langle x \rangle = a\langle n \rangle = \frac{a}{\tau} t (p-q) = \frac{a}{\tau} t(2p-1) = vt$$

$\epsilon = p-q$ "bias" $v = \frac{a}{\tau}\epsilon$

$$\langle (n-\langle n\rangle)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2$$

$$\langle n_\rightarrow^2 \rangle = \sum_{n_\rightarrow=0}^{N} n_\rightarrow^2 \frac{N!}{n_\rightarrow!(N-n_\rightarrow)!} p^{n_\rightarrow} q^{N-n_\rightarrow}$$

$$= \frac{\partial^2}{\partial\mu^2} \underbrace{\sum_{n_\rightarrow=0}^{N} \frac{N!}{n_\rightarrow!(N-n_\rightarrow)!} p^{n_\rightarrow} q^{N-n_\rightarrow} e^{\mu n_\rightarrow}}_{\text{Generating function}} \Big|_{\mu\to 0}$$

$$= q^N \frac{\partial^2}{\partial\mu^2} \sum_{n_\rightarrow=0}^{N} \frac{N!}{n_\rightarrow!(N-n_\rightarrow)!} \left(\frac{p}{q} e^\mu\right)^{n_\rightarrow} \Big|_{\mu\to 0}$$

$$= q^N \frac{\partial}{\partial\mu} \left(1-\frac{p}{q}e^\mu\right)^N \Big|_{\mu\to 0}$$

$$= q^N \frac{\partial}{\partial\mu} \left[N\frac{p}{q}e^\mu \left(1+\frac{p}{q}e^\mu\right)^{N-1}\right] \Big|_{\mu\to 0}$$

$$= pN\left[1+p(N-1)\right]$$

$$\mathrm{var}(n) = \langle \delta n^2 \rangle = \langle (n-\langle n\rangle)^2 \rangle$$

$$= \langle [2n_\rightarrow - N - (2\langle n_\rightarrow\rangle - N)]^2 \rangle$$

$$= 4\langle (n_\rightarrow - \langle n_\rightarrow \rangle)^2 \rangle$$

$$= 4\, \mathrm{var}(n_\rightarrow)$$

$$\mathrm{var}(n_\rightarrow) = \langle n_\rightarrow^2 \rangle - \langle n_\rightarrow \rangle^2$$

$$= pN + (pN)^2 - p^2 N - (pN)^2$$

$$= Np(1-p)$$

$p-q=\varepsilon$ $\quad p = \frac{\varepsilon+1}{2}$, $q = \frac{1-\varepsilon}{2}$

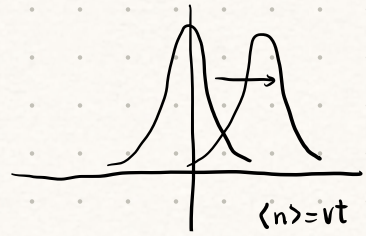
$$\mathrm{var}(n_\rightarrow) = \frac{1}{4}N(1-\varepsilon^2)$$

$$\mathrm{var}(n) = N(1-\varepsilon)^2$$

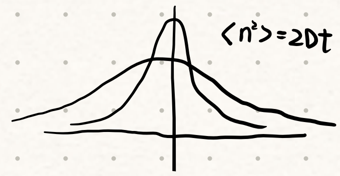
$$\langle (\delta x)^2 \rangle = \frac{t}{\tau} a^2 (1-\varepsilon^2) = 2Dt(1-\varepsilon^2)$$

$$\begin{cases} 2Dt & \epsilon = 0 \\ 0 & \epsilon = 1 \end{cases}$$

$p=1, q=0$



$p=q=\frac{1}{2}$



补充：关于Generation function

本质上是凑二项式+求导 另一种形式：

$$\langle n_\rightarrow \rangle = \sum_{n_\rightarrow=0}^{N} \frac{n_\rightarrow N!\, p^{n_\rightarrow} q^{N-n_\rightarrow}}{(N-n_\rightarrow)!\, n_\rightarrow!}$$

$$(p+q)^N = \sum_{m=0}^{N} \frac{N!\, p^{n_\rightarrow} q^{N-n_\rightarrow}}{(N-n_\rightarrow)!\, n_\rightarrow!}$$

$\frac{\partial}{\partial p}$ $\times p$

$$Np(\overset{1}{p+q})^{N-1} = \sum_{n_\rightarrow=0}^{N} \frac{n_\rightarrow N!\, p^{n_\rightarrow} q^{N-n_\rightarrow}}{(N-n_\rightarrow)!\, n_\rightarrow!}$$

$\Rightarrow \langle n_\rightarrow \rangle = Np$

$$Np(p+q)^{N-1} = \sum_{n_\rightarrow=0}^{N} \frac{n_\rightarrow N!\, p^{n_\rightarrow} q^{N-n_\rightarrow}}{(N-n_\rightarrow)!\, n_\rightarrow!}$$

$\frac{\partial}{\partial p}$ $\times p$

$$Np(\overset{1}{p+q})^{N-2}(pN+q) = \sum_{n_\rightarrow=0}^{N} \frac{n_\rightarrow^2 N!\, p^{n_\rightarrow} q^{N-n_\rightarrow}}{(N-n_\rightarrow)!\, n_\rightarrow!}$$

$$\langle n_\rightarrow^2 \rangle = Np(pN+q)$$

$$\mathrm{var}(n_\rightarrow) = \langle n_\rightarrow^2 \rangle - \langle n_\rightarrow \rangle^2 = Npq$$

Arbitary random walk

$P = P(x)$ $\quad q = q(x)$ $\quad q(x)+P(x)=1$

$\cdots$ $x-a$ $\quad x$ $\quad x+a$ $\cdots$

Back to the symmetric RW $p(x)=q(x)=\frac{1}{2}$ →

particle is at x at time t

$$\underbrace{P(x,t)}_{P(x,t-\tau)+\tau\frac{\partial}{\partial t}P(x,t-\tau)} = \frac{1}{2}P(x-a,t-\tau) + \frac{1}{2}P(x+a,t-\tau)$$

$$\tau\frac{\partial}{\partial t}P(x,t-\tau) = \frac{1}{2}P(x-a,t-\tau) + \frac{1}{2}P(x+a,t-\tau) - P(x,t-\tau)$$

$t-\tau \to t$

Master equation

$$\frac{\partial}{\partial t}P(x,t) = \frac{1}{2\tau}\left[P(x-a,t)+P(x+a,t)-2P(x,t)\right]$$

$$\frac{\partial}{\partial t}P(n,t) = \frac{1}{2\tau}\left(P(n-1,t)+P(n+1,t)-2P(n,t)\right)$$

本质是

$$+\begin{cases} x-a \to x \\ x+a \to x \end{cases} \qquad -\begin{cases} x \to x-a \\ x \to x+a \end{cases}$$

$a \ll 1$

$$P(x\pm a) = P(x) \pm a\frac{\partial}{\partial x}P(x,t) + \frac{1}{2}a^2\frac{\partial^2}{\partial x^2}P(x,t)$$

$$\frac{\partial P}{\partial t} = \frac{1}{2\tau}\left[P + aP_x + \frac{1}{2}a^2 P_{xx} + P(x,t) - aP_x + \frac{1}{2}a^2 P_{xx} - 2P\right]$$

$$= \underbrace{\left(\frac{a^2}{2\tau}\right)}_{D} \frac{\partial^2}{\partial x^2}P(x,t)$$

Fokker-Planck Equation

Diffusion equation

special case $v=0$

test by plugging in

$$P(x,t|0,0) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right)$$

The top $\sim\frac{3}{4}$ of this page shows how to calculate the first and second moments of a binomial distribution. Now let's see those of Poisson distribution and geometric distribution.

Poisson

$$P(x=m;\lambda)=\frac{\lambda^m e^{-\lambda}}{m!} \quad (m=0,1,2,\cdots)$$

$$\sum_{m=0}^{\infty}\frac{\lambda^m e^{-\lambda}}{m!}=e^{-\lambda}\sum_m \frac{\lambda^m}{m!}$$
$$=e^{-\lambda}e^{\lambda}=1$$

$$\langle x\rangle=\sum_m m\frac{\lambda^m e^{-\lambda}}{m!}$$
$$=e^{-\lambda}\sum_m m\frac{\lambda^m}{m!}e^{\mu m}\Big|_{\mu\to 0}$$
$$=e^{-\lambda}\frac{\partial}{\partial\mu}\sum_m\frac{\lambda^m e^{\mu m}}{m!}\Big|_{\mu\to 0}$$
$$=e^{-\lambda}\frac{\partial}{\partial\mu}e^{\lambda e^{\mu}}\Big|_{\mu\to 0}$$
$$=e^{-\lambda}(\lambda e^{\mu})e^{\lambda e^{\mu}}\Big|_{\mu\to 0}$$
$$=e^{-\lambda}\lambda e^{\lambda}$$
$$=\lambda$$

$$\langle x^2\rangle=\sum_m m^2\frac{\lambda^m e^{-\lambda}}{m!}$$
$$=e^{-\lambda}\frac{\partial^2}{\partial\mu^2}\sum_m\frac{\lambda^m e^{\mu m}}{m!}\Big|_{\mu\to 0}$$
$$=e^{-\lambda}\frac{\partial^2}{\partial\mu^2}e^{\lambda e^{\mu}}\Big|_{\mu\to 0}$$
$$=e^{-\lambda}\lambda e^{\mu}e^{\lambda e^{\mu}}(\lambda e^{\mu}+1)\Big|_{\mu\to 0}$$
$$=e^{-\lambda}\lambda e^{\lambda}(\lambda+1)$$
$$=\lambda^2+\lambda$$

$$\mathrm{Var}(x)=\langle x^2\rangle-\langle x\rangle^2$$
$$=(\lambda^2+\lambda)-\lambda^2$$
$$=\lambda$$

Geometric

$$P(x=m;p)=(1-p)^{m-1}p \quad (m=1,2,\cdots)$$

$$\sum_{m=1}^{\infty}(1-p)^{m-1}p=p\sum_m(1-p)^{m-1}$$
$$=p\lim_{m\to\infty}1\cdot\frac{1-(1-p)^m}{1-(1-p)}=1$$

$$\langle x\rangle=\sum_m m(1-p)^{m-1}p=p\sum_m m(1-p)^{m-1}e^{\mu m}\Big|_{\mu\to 0}$$
$$=p\frac{\partial}{\partial\mu}\sum_m(1-p)^{m-1}e^{\mu m}\Big|_{\mu\to 0}=p\frac{\partial}{\partial\mu}\lim_{m\to\infty}e^{\mu}\frac{1-[(1-p)e^{\mu}]^m}{1-(1-p)e^{\mu}}\Big|_{\mu\to 0}$$
$$=p\frac{\partial}{\partial\mu}e^{\mu}\frac{1}{1-(1-p)e^{\mu}}\Big|_{\mu\to 0}=p\frac{e^{\mu}[1-(1-p)e^{\mu}+(1-p)e^{\mu}]}{[1-(1-p)e^{\mu}]^2}\Big|_{\mu\to 0}$$
$$=p\frac{e^{\mu}}{[1-(1-p)e^{\mu}]^2}\Big|_{\mu\to 0}=p\frac{1}{[1-(1-p)]^2}$$
$$=\frac{1}{p}$$

$$\langle x^2\rangle=\sum_m m^2(1-p)^{m-1}p=p\sum_m m^2(1-p)^{m-1}e^{\mu m}\Big|_{\mu\to 0}$$
$$=p\frac{\partial^2}{\partial\mu^2}\sum_m(1-p)^{m-1}e^{\mu m}\Big|_{\mu\to 0}=p\frac{\partial^2}{\partial\mu^2}\lim_{m\to\infty}e^{\mu}\frac{1-[(1-p)e^{\mu}]^m}{1-(1-p)e^{\mu}}\Big|_{\mu\to 0}$$
$$=p\frac{\partial^2}{\partial\mu^2}e^{\mu}\frac{1}{1-(1-p)e^{\mu}}\Big|_{\mu\to 0}=p\frac{\partial}{\partial\mu}\frac{e^{\mu}}{[1-(1-p)e^{\mu}]^2}\Big|_{\mu\to 0}$$
$$=p\frac{[1-(1-p)^2e^{2\mu}]e^{\mu}}{[1-(1-p)e^{\mu}]^4}\Big|_{\mu\to 0}=p\frac{1-(1-p)^2}{[1-(1-p)]^4}$$
$$=\frac{2-p}{p^2}$$

$$\mathrm{Var}(x)=\langle x^2\rangle-\langle x\rangle^2$$
$$=\frac{2-p}{p^2}-\frac{1}{p^2}$$
$$=\frac{1-p}{p}$$