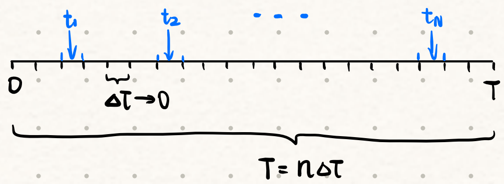
Suppose the rate of a event happens is $r(t)$, which is inhomogeneous.

The probability of

During $\tau\in(0,T)$, N events happen at $(t_1, t_2, \cdots, t_N)$ is $P[\{t_i\}|r(\tau)]$



$$P[\{t_i\}|r(\tau)](\Delta\tau)^N = \underbrace{\prod_{n\neq i}[1-r(t_n)\Delta\tau]}_{\text{not happen}}\ \underbrace{\prod_{i=1}^{N}[r(t_i)\Delta\tau]}_{\text{happen}}$$

put all "happen" into "not happen"

$$= \underbrace{\prod_{n}[1-r(t_n)\Delta\tau]}_{A}\ \underbrace{\prod_{i=1}^{N}\left[\frac{r(t_i)\Delta\tau}{1-r(t_i)\Delta\tau}\right]}_{B}$$

$$A \equiv \prod_n[1-r(t_n)\Delta\tau] = \exp\left(\sum_n \ln[1-r(t_n)\Delta\tau]\right)$$

$$= \frac{1}{N!}\exp\left(\sum_n[-r(t_n)\Delta\tau] - \frac{1}{2}\sum_n[-r(t_n)\Delta\tau]^2 + \cdots\right)$$

$$\approx \exp[-\int dt\ r(t) - \frac{1}{2}\Delta\tau\int dt\ r^2(t) + \cdots]$$

$$B \equiv \prod_{i=1}^{N}\frac{r(t_i)\Delta\tau}{1-r(t_i)\Delta\tau} = (\Delta\tau)^N\left[\prod_{i=1}^{N} r(t_i)\right]\left\{\prod_{j=1}^{N}[1-r(t_j)\Delta\tau]\right\}^{-1}$$

$$= (\Delta\tau)^N\left[\prod_{i=1}^{N} r(t_i)\right]\left[1 - \sum_{j=1}^{N} r(t_j)\Delta\tau + \cdots\right]$$

$$P[\{t_i\}|r(\tau)](\Delta\tau)^N = AB \rightarrow (\Delta\tau)^N \exp\left[-\int_0^T dt\ r(t)\right]\prod_{i=1}^{N} r(t_i)$$

$$P[\{t_i\}|r(\tau)] = \exp\left[-\int_0^T dt\ r(t)\right]\prod_{i=1}^{N} r(t_i)$$

Check the normalization: sum the probability of

During $\tau\in(0,T)$, $N=0,1,\cdots,\infty$ happen should be 1

$$Z \equiv \sum_{N=0}^{\infty}\frac{1}{N!}\int_0^T dt_1\int_0^T dt_2\cdots\int_0^T dt_N\ P[\{t_i\}|r(t)]$$

$$= \sum_{N=0}^{\infty}\frac{1}{N!}\int_0^T dt_1\int_0^T dt_2\cdots\int_0^T dt_N\ \underbrace{\exp\left[-\int_0^T dt\ r(t)\right]}_{\text{not depend on }\{t_i\}}\prod_{i=1}^{N} r(t_i)$$

$$= \exp\left[-\int_0^T dt\ r(t)\right]\sum_{N=0}^{\infty}\frac{1}{N!}\int_0^T dt_1\cdots\int_0^T dt_N\ r(t_1)\cdots r(t_N)$$

$$= \exp\left[-\int_0^T dt\ r(t)\right]\sum_{N=0}^{\infty}\frac{1}{N!}\int_0^T dt_1\ r(t_1)\int_0^T dt_2\ r(t_2)\cdots\int_0^T dt_N\ r(t_N)$$

$$= \underbrace{\exp\left[-\int_0^T dt\ r(t)\right]\sum_{N=0}^{\infty}\frac{1}{N!}\left[\int_0^T dt\ r(t)\right]^N}_{\text{series expansion of exponential function}}$$

$$= \exp\left[-\int_0^T dt\ r(t)\right]\exp\left[\int_0^T dt\ r(t)\right]$$

$$= 1$$

Distribution of counting N events during $\tau\in(0,T)$

take the full distribution $P[\{t_i\}|r(\tau)]$ and sum over all possible arriving times

we denote it as $P(N|\ ?\ )$ since we'll see its shape depends only on $\langle N\rangle$,

so we can write $P(N|\langle N\rangle)$

$$P(N|\ ?\ ) = \frac{1}{N!}\int_0^T dt_1\cdots\int_0^T dt_N\ P[\{t_i\}|r(\tau)]$$

$$= \frac{1}{N!}\exp\left[-\underbrace{\int_0^T dt\ r(t)}_{Q}\right]\left[\int_0^T dt\ r(t)\right]^N$$

$$= P(N|Q)$$

$Z = \sum_{n=0}^{\infty} P(N|\langle N\rangle)$ and check the pink line

$$Q \equiv \int_0^T dt\ r(t) \Longrightarrow P(0|\langle N\rangle) = \exp\left[-\int_0^T dt\ r(t)\right] = \exp(-Q)$$

$$\langle N\rangle \equiv \sum_{n=0}^{\infty} P(N|Q)\,N$$

$$= \sum_{n=0}^{\infty}\frac{1}{N!}\exp(-Q)Q^N N$$

$$= \exp(-Q)\sum_{n=0}^{\infty}\frac{1}{N!}Q^N N$$

$$= \exp(-Q)\sum_{n=0}^{\infty}\frac{1}{N!}Q\frac{\partial}{\partial Q}Q^N$$

$$\langle N^2\rangle \equiv \sum_{n=0}^{\infty} P(N|Q)\,N^2$$

$$= \sum_{n=0}^{\infty} N^2\exp(-Q)\frac{1}{N!}Q^N$$

$$= \exp(-Q)\sum_{n=0}^{\infty}\frac{1}{N!}N^2 Q^N$$

$$= \exp(-Q)\sum_{n=0}^{\infty}\frac{1}{N!}\left[Q^2\frac{\partial^2}{\partial Q^2}Q^N + Q\frac{\partial}{\partial Q}Q^N\right]$$

$$\frac{\partial^2}{\partial Q^2}Q^N = N(N-1)Q^{N-2}$$

$$Q^2\frac{\partial^2}{\partial Q^2}Q^N = (N^2-N)Q^N$$

$$N^2 Q^N = Q^2\frac{\partial^2}{\partial Q^2}Q^N + Q\frac{\partial}{\partial Q}Q^N$$

$= \exp(-Q)\, Q \frac{\partial}{\partial Q} \sum_{n=0}^{\infty} \frac{1}{N!} Q^N$

$= \exp(-Q)\, Q \frac{\partial}{\partial Q} \exp(Q)$

$= \exp(-Q)\, Q \exp(Q)$

$= Q$

$P(N|Q) = P(N|\langle N\rangle) = \exp(-\langle N\rangle) \frac{\langle N\rangle^N}{N!}$

$= \exp(-Q)\, Q^2 \frac{\partial^2}{\partial Q^2} \sum_{n=0}^{\infty} \frac{1}{N!} Q^N + \exp(-Q)\, Q \frac{\partial}{\partial Q} \sum_{n=0}^{\infty} \frac{1}{N!} Q^N$

$= \exp(-Q)\, Q^2 \frac{\partial^2}{\partial Q^2} \sum_{n=0}^{\infty} \frac{1}{N!} Q^N + \exp(-Q)\, Q \frac{\partial}{\partial Q} \exp(Q)$

$= \exp(-Q)\, Q^2 \exp(Q) + \exp(-Q)\, Q \exp(Q)$

$= Q^2 + Q$

$= \langle N\rangle^2 + \langle N\rangle$

$\langle (\delta N)^2\rangle \equiv \langle N^2\rangle - \langle N\rangle^2 = \langle N\rangle$ Nice :-)

Conclusion: the variance of the count for a Poisson process is equal to the mean count. The standard deviation of the Poisson distribution is the square root of the mean, and the square root of N law is one of the most improtant intuitions about the statistics of counting independent events.

Time interval between events

The probability that no events in $[t, t+\tau)$ is

$P(0) = \exp\left[-\int_t^{t+\tau} dt'\, r(t')\right]$

Then two events happens at t and $t+\tau$ is

$P(t, t+\tau) = r(t) \exp\left[-\int_t^{t+\tau} dt'\, r(t')\right] r(t+\tau)$

$P_2(\tau) = \left\langle r(t) \exp\left[-\int_t^{t+\tau} dt'\, r(t')\right] r(t+\tau)\right\rangle_r$

↑ two events

If $r(t) = r$ (homogeneous). $P(t, t+\tau) = r^2 e^{-r\tau}$

Given an event happens at t, the conditional probability becomes

$P(\tau) = r e^{-r\tau}$