the k-th sample moment : $M_k = \frac{1}{N}\sum_{i=1}^{N} X_i^k$.

for i.i.d. X_i, $E[M_k] = \frac{1}{N}\sum_{i=1}^{N} E[X_i^k] = E[X^k]$.

$$E[M_k M_j] = E\left[\left(\frac{1}{N}\sum_{i=1}^{N} X_i^k\right)\left(\frac{1}{N}\sum_{l=1}^{N} X_l^j\right)\right] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{l=1}^{N} E[X_i^k X_l^j].$$

- when $i=l$, $E[X_i^k X_l^j] = E[X_i^{k+j}] = E[X^{k+j}]$;
- when $i \neq l$, i.i.d. $\Rightarrow$ X_i and X_j are independent,

  $E[X_i^k X_l^j] = E[X_i^k]E[X_l^j] = E[X^k]E[X^j]$.

$$E[M_k M_j] = \frac{1}{N^2}\left(N\,E[X^{k+j}] + N(N-1)\,E[X^k]\,E[X^j]\right)$$

$$\begin{aligned} Cov(M_k, M_j) &= E[M_k M_j] - E[M_k]\,E[M_j] \\ &= \frac{1}{N^2}\left(N\,E[X^{k+j}] + N(N-1)\,E[X^k]\,E[X^j]\right) - E[X^k]\,E[X^j] \\ &= \frac{1}{N}\left(E[X^{k+j}] - E[X^k]\,E[X^j]\right) \end{aligned}$$